Proceedings of the 2023 International Conference on Mathematical Physics and Computational Simulation
DOI: 10.54254/2753-8818/10/20230324

Lobachevsky's formula and its applications to several definite integrals

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Abstract. The definite integral is a fundamental concept in calculus that has many applications in various fields such as physics, engineering, and economics. However, integration can be difficult and requires a variety of skills such as substitutions and partial integration. In this paper, Lobachevsky's formula is explored, which provides a new way to evaluate definite integrals. It should be noted that Lobachevsky's formula can only be applied in specific cases where the integrand is even and π-periodic. However, it is demonstrated to be an effective method in these cases. In this paper, the proof of the theorem is given, and a variety of examples are solved by virtue of this method. Hence, this paper may serve as a reference for relevant research in the field of calculus and provide insights into the applications of Lobachevsky's formula.

Keywords: Lobachevsky's formula, definite integral, improper integral, calculus.

1. Introduction

Calculus is a branch of mathematics that deals with the study of continuous change and motion. The history of calculus dates back to ancient Greece, where the Greeks used the method of exhaustion to calculate areas and volumes [1]. However, it wasn't until the 17th century that calculus was developed into a formal mathematical discipline by Isaac Newton and Gottfried Wilhelm Leibniz. As one of the main focuses of calculus, integrals have many applications in various fields such as physics and statistics. For instance, the Gaussian integral, which is about the normalization of the normal distribution function, is a definite integral that appears widely in probability theory and physics. The Fresnel integrals are another example of definite integrals. Originating in optics, they were introduced to calculate the diffraction pattern produced by a rectangular aperture. They have been applied in the design of highways and railways to create smooth transitions between curves and straight lines [2].

A variety of methods can be applied to compute definite integrals. One example is Feynman's parameterization trick, which can solve a lot of problems that seems impossible to solve at first glance. This powerful technique involves parameterizing the integrand and differentiating the integral with respect to the parameter to obtain a differential equation. Another example is the series method, which requires expressing the integrand as a power series and then integrating term by term to obtain an infinite series representation of the integral. Although it could be challenging to get explicit results, this method is still effective at solving integrals which are difficult or impossible to solve otherwise [3]. The residue theorem is also a typical method. By finding a complex analytic function closely connected to the integrand and applying the residue theorem to compute its integral along some closed contours, it's possible to deduce the value of the desired definite integral. However, it's not always easy to find such functions.

This article introduces Lobachevsky's Formula which is an unconventional and powerful tool for solving definite integrals involving trigonometric functions. It is named after Nikolai Ivanovich Lobachevsky, who was a Russian mathematician and geometer known for his work on non-Euclidean geometry [4]. The formula is unique because it uses a sinc function as a weight instead of the usual exponential function used in other methods of integral calculation. Although the condition of the theorem that the function should be even and π-periodic is not always easily satisfied, when it can be applied, it can significantly simplify the calculation of integrals. This article provides a comprehensive overview of Lobachevsky's integral formula. Section 2 of this article provides both the formula and a proof. In Section 3, a variety of examples and applications of the formula are presented, while Section 4 concludes this article.

2. Lobachevsky's formula

A proof of Lobachevsky's formula is provided here.

Theorem. If a function $f\colon \mathrm{R} \to \mathrm{R}$ satisfies $f(x) = f(\pi + x) = f(\pi - x)$ for all $x \in \mathrm{R}$, and f is Riemann integrable on $\left[0, \frac{\pi}{2}\right]$, then the following formula holds [5]:

$$\int_0^\infty \frac{\sin x}{x} f(x)\,dx = \int_0^{\frac{\pi}{2}} f(x)\,dx. \tag{1}$$

Proof. First rewrite the integration as

$$\int_0^\infty \frac{\sin x}{x} f(x)\,dx = \sum_{k=0}^{\infty} \int_{k\pi}^{\left(k+\frac{1}{2}\right)\pi} \frac{\sin x}{x} f(x)\,dx + \sum_{k=1}^{\infty} \int_{\left(k-\frac{1}{2}\right)\pi}^{k\pi} \frac{\sin x}{x} f(x)\,dx. \tag{2}$$

Now make substitutions to see that

$$\int_{k\pi}^{\left(k+\frac{1}{2}\right)\pi} \frac{\sin x}{x} f(x)\,dx = \int_0^{\frac{\pi}{2}} \frac{(-1)^k \sin x}{x + k\pi} f(x)\,dx, \tag{3}$$

$$\int_{\left(k-\frac{1}{2}\right)\pi}^{k\pi} \frac{\sin x}{x} f(x)\,dx = \int_0^{\frac{\pi}{2}} \frac{(-1)^k \sin x}{x - k\pi} f(x)\,dx. \tag{4}$$

Plug (3) and (4) into (2) and the equation turns into

$$\int_0^\infty \frac{\sin x}{x} f(x)\,dx = \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \left(\frac{1}{x} + \sum_{k=1}^{n} (-1)^k \left(\frac{1}{x - k\pi} + \frac{1}{x + k\pi}\right)\right) \sin x\, f(x)\,dx. \tag{5}$$

Denote the first part of the integrand as

$$U_n(x) := \frac{1}{x} + \sum_{k=1}^{n} (-1)^k \left(\frac{1}{x - k\pi} + \frac{1}{x + k\pi}\right) = \sum_{k=-n}^{n} \frac{(-1)^k}{x + k\pi}. \tag{6}$$

Fourier series can be used to calculate $U_n(x)$. Take $\alpha \in [-\pi, \pi]$, $y \in R$ and consider the Fourier series of $\cos y\alpha$ about α. Since this is an even function, all the coefficients of $\sin k\alpha$ disappear. Thus

$$\cos y\alpha = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\alpha. \tag{7}$$

where the coefficients are

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \cos(y\alpha)\,d\alpha = \frac{2 \sin \pi y}{\pi y}, \tag{8}$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} \cos(y\alpha) \cos(k\alpha)\,d\alpha = (-1)^k \frac{\sin(\pi y)}{\pi} \left(\frac{1}{y + k} + \frac{1}{y - k}\right), k \geq 1. \tag{9}$$

Now take $\alpha = 0$ in (7), it is obtained that

$$1 = \sin(\pi y) \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\pi y + k\pi} = \sin(\pi y) \lim_{n\to\infty} U_n(\pi y)\,. \tag{10}$$

For all $x \in \left(0, \frac{\pi}{2}\right)$, taking $\pi y = x$ immediately shows that $\lim_{n\to\infty} U_n(x) = \frac{1}{\sin x}$. Finally, the dominated convergence theorem is applied to (5) to get the desired result [6]:

$$\int_0^{\infty} \frac{\sin x}{x} f(x) dx = \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} U_n(x) \sin(x) f(x) dx = \int_0^{\frac{\pi}{2}} f(x) dx\,. \tag{11}$$

The only thing left undone is to check the condition of the Dominated Convergence Theorem, namely, the requirement that there exist a real number M such that

$$|U_n(x) \sin(x)\, \mathrm{f}(x)| \le M \tag{12}$$

holds for all $x \in \left(0, \frac{\pi}{2}\right)$ and positive integer n. Since the fact that $f(x)$ is Riemann integrable already implies that it's bounded [7], if the uniform boundedness of $U_n(x)\sin(x)$ is obtained then the proof would be complete. For this purpose, using the Alternating Series Approximation Theorem, the following estimations holds for all $x \in \left(0, \frac{\pi}{2}\right)$ and positive integer n:

$$\left|\sum_{k>n} \frac{(-1)^k}{x + \pi k}\right| \le \left|\frac{1}{x + (1+n)\pi}\right| \le \frac{1}{n\pi}, \tag{13}$$

$$\left|\sum_{k>n} \frac{(-1)^k}{x - \pi\, k}\right| \le \left|\frac{1}{x - (1+n)\pi}\right| \le \frac{1}{n\pi}. \tag{14}$$

By (6) and (10), it follows that

$$|U_n(x) \sin(x)| \le \left|1 + \frac{2}{n\pi}\right| \le 2, \tag{15}$$

and the proof of the theorem is now completed.

3. Applications

3.1. *Example I*

Find this integral

$$I = \int_0^{\infty} \frac{\sin x}{x + x\cos^2 x} dx. \tag{16}$$

Set $\mathrm{f}(x) = \frac{1}{1+\cos^2 x}$, apparently $f(x)$ satisfies the conditions of Lobachevsky's theorem. Apply the theorem to obtain

$$I = \int_0^{\infty} f(x) dx = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos^2 x}. \tag{17}$$

The substitution $\mathrm{t} = \tan x$ gives

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1 + \cos^2 x} dt = \int_0^{\infty} \frac{dt}{t^2 + 2}. \tag{18}$$

Subsequently,

$$I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\Big|_0^{\infty} = \frac{\pi}{2\sqrt{2}}. \tag{19}$$

3.2. *Example II*

Find this integral

$$I = \int_0^{\infty} \frac{|\sin x| \sin x}{x} dx\,. \tag{20}$$

Set $\mathrm{f}(x) = |\sin x|$ and apply the theorem, one has

$$I = \int_0^{\frac{\pi}{2}} |\sin x| dx = 1. \tag{21}$$

3.3. *Example III*

Find this integral

$$\mathrm{I} = \int_0^{\infty} \frac{\tan^{-1}(\sin x)}{x} dx\,. \tag{22}$$

This integral can be solved with the help of Feynman's parameterization trick [8], to be specific, by introducing a parameter $a > 0$ in the following way:

$$\mathrm{I}(a) = \int_0^{\infty} \frac{\tan^{-1}(a \sin x)}{x} dx\,. \tag{23}$$

Now set $f(x) = \frac{\tan^{-1}(a\sin x)}{\sin x}$, it is easy to see that $f(x)$ is even and π-periodic. Apply Lobachevsky's theorem and the integral turns into

$$\mathrm{I}(a) = \int_0^{\infty} f(x)\frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) dx = \int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(a \sin x)}{\sin x} dx\,. \tag{24}$$

Now make the substitution $\mathrm{u} = \sin \mathrm{x}$, $dx = \frac{du}{\sqrt{1-u^2}}$ to obtain

$$\mathrm{I}(a) = \int_0^{1} \frac{\tan^{-1}(a \sin x)}{u\sqrt{1-u^2}} du\,. \tag{25}$$

It is time to take a derivative with respect to the parameter a, i.e.,

$$I'(a) = \int_0^{1} \frac{du}{(1+(au)^2)\sqrt{1-u^2}}. \tag{26}$$

Now make another substitution $\mathrm{u} = \cos\theta$, $\mathrm{du} = -\sin\theta\, \mathrm{d}\theta$, and it becomes

$$\mathrm{I}'(a) = \int_{\frac{\pi}{2}}^{0} \frac{-\sin\theta\, d\theta}{(1+(a\cos\theta)^2)\sin\theta} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+(a\cos\theta)^2}. \tag{27}$$

This is the same integral as in Section 3.1 except for a minor difference in constants, thus by the same method, one gets

$$I'(a) = \frac{\pi}{2\sqrt{1+a^2}}. \tag{28}$$

Finally, integrate this to find $I(a)$:

$$I(a) = I(0) + \int_0^{a} I'(t) dt = \frac{\pi}{2}\sinh^{-1} a. \tag{29}$$

The original integral is just $\mathrm{I}(1) = \frac{\pi}{2}\sinh^{-1} 1$.

3.4. *Variant version of the theorem*

Theorem. If a function $f\colon R \to R$ satisfies $\mathrm{f}(\pi - x) = \mathrm{f}(\pi + x) = -\mathrm{f}(x)$ for all $x \in R$, and that f is Riemann integrable on $\left[0, \frac{\pi}{2}\right]$, then the following formula holds:

$$\int_0^{\infty} \frac{\sin x}{x} f(x) dx = \int_0^{\frac{\pi}{2}} f(x) \cos x\, dx\,. \tag{30}$$

Note that the condition still means that $f(x)$ is even, but it is no longer π-periodic. Adding a π to its parameter changes the sign of its value.

Proof. Following the same lines as the proof in Section 2, the integral can be rewritten as

$$\int_0^{\infty} \frac{\sin x}{x} f(x) dx = \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \left(\sum_{k=-n}^{n} \frac{1}{x + k\pi} \right) \sin x\, f(x) dx\,. \tag{31}$$

Set

$$V_n(x) := \sum_{k=-n}^{n} \frac{1}{x+k\pi} \tag{32}$$

Now take $\alpha = \pi$ and $\pi y = x$ in (6), it shows that

$$\cot(x) = \lim_{x\to\infty} V_n(x). \tag{33}$$

In contrast to the alternating series behaviour of $U_n(x)$, $V_n(x)$ is strictly decreasing, because for all positive integer n,

$$\frac{1}{x+n\pi} + \frac{1}{x-n\pi} = \frac{2x}{x^2-n^2\pi^2} < 0. \tag{34}$$

Also note that for $n = 1$, $V_1(x)\sin(x) = \frac{\sin(x)}{x}$ has a limit of 1 as $x \to 0$ and thus it is bounded over $\left(0, \frac{\pi}{2}\right)$. Therefore, the uniform boundness of $V_n(x)\sin(x)$ automatically follows from the fact that $V_n(x) \to \cot(x)$ as $\mathrm{n} \to \infty$. So, by the dominated convergence theorem [9], the desired result (30) derives from (31).

3.5. *Example IV*

Find this integral

$$I = \int_0^\infty \frac{\sin(\tan(x))}{x} dx\,. \tag{35}$$

First set $\mathrm{f}(x) = \frac{\sin(\tan(x))}{\sin(x)}$. Although it doesn't satisfy the conditions of the original theorem in Section 2, it instead satisfies the condition in the Section 3.4, because $f(x)$ is even and $f(x+\pi) = -f(x)$ holds for all x. Apply the result to see that

$$\mathrm{I} = \int_0^\infty f(x)\cos(x)\,\mathrm{d}\,x = \int_0^\infty \frac{\sin(\tan(x))}{\tan(x)} dx\,. \tag{36}$$

Now make the substitution $\mathrm{t} = \tan(x)$ to see that

$$I = \int_0^\infty \frac{\sin t}{t(1+t^2)} dt = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin t}{t(1+t^2)} dt\,. \tag{37}$$

This integral can be calculated using the residue theorem. First notice that

$$\frac{\sin t}{t(1+t^2)} = \frac{\sin t}{t} - \frac{t\sin t}{t^2+1}. \tag{38}$$

Plug (37) into (28) to get

$$I = \int_0^\infty \frac{\sin t}{t} dt - \frac{1}{2}\int_{-\infty}^\infty \frac{t\sin t}{t^2+1} dt = \frac{\pi}{2} - \frac{1}{2} I_2, \tag{39}$$

where

$$\mathrm{I}_2 = \int_{-\infty}^\infty \frac{t\sin t}{t^2+1} dt = \mathrm{Im}\int_{\mathrm{R}} \frac{ze^{iz}}{z^2+1} dz\,. \tag{40}$$

Denote the integrand as $g(z)$. It has two poles of order 1, respectively i and $-i$. Consider the contour $C = R + L$, where R is the real axis and L is the counterclockwise semicircle path above the real axis of radius $r > 0$. Only i is enclosed in this contour. Thus, by the residue theorem,

$$\int_C g(z)dz = \int_R g(z)dz + \int_L g(z)dz = 2\pi\mathrm{i}\mathrm{Res}(g; i) = 2\pi\mathrm{i}\cdot \left.\frac{ze^{iz}}{z+i}\right|_{z=i} = \frac{\pi\mathrm{i}}{e}. \tag{41}$$

Finally, by Jordan's lemma [10], the following inequality holds:

$$\left|\int_L g(z)dz\right| \le \pi \max_{0\le\theta\le\pi}\left|\frac{re^{i\theta}}{(re^{i\theta})^2+1}\right|. \tag{42}$$

which means the integral $\int_L g(z)dz$ vanishes as $r \to \infty$. Therefore

DOI: 10.54254/2753-8818/10/20230324

$$\int_R g(z)dz = \frac{\pi i}{e}, \tag{43}$$

$$I_2 = \operatorname{Im} \int_R g(z)dz = \frac{\pi}{e}. \tag{44}$$

Hence, the value of the desired integral is found:

$$I = \frac{\pi}{2} - \frac{1}{2} I_2 = \frac{\pi}{2} - \frac{\pi}{2e}. \tag{45}$$

4. Conclusion

This article explores the Lobachevsky's formula, which is a remarkable result that facilitates the calculation of certain definite integrals involving trigonometric functions. In this article, a proof of the formula based on Fourier expansion is presented, and how it can be applied to various examples is shown. It is revealed that for the integrals in specific forms which the theorem can be applied, the complexity of calculation can be significantly reduced. Some examples shown in this article also demonstrate how it can be used along with other methods, such as Feynman's parametrization trick and the residue theorem, to solve complicated definite integrals. Moreover, the formula reveals some connection between infinite series and integrals. Despite the strict restrictions that the integrand must satisfy in this formula, by using the theory of infinite series it can be generalized. The author encourages further research on this topic, as Lobachevsky's formula is an interesting and useful result that deserves attention and appreciation.

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