DOI: 10.54254/2753-8818/5/20230307

Complex Analysis and Residue Theorem

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Abstract. The study of the properties of analytical functions is described as complex analysis. The residue theorem is an important conclusion in complex analysis. This paper introduces the origin of imaginary numbers from Cardano Formula defines the conversion of complex number formats from the Euler Theorem, and proves the intermediate theorem Cauchy Integral Formula before reaching our final conclusion and the goal of the paper, Residue Theorem. The name of the theorem comes from the concept of residue, which is defined using a function's Laurent series. We could then derive the Residue Theorem from the Cauchy Integral Theorem, also called the Cauchy-Goursat Theorem. We will be able to formalize our prior, ad hoc method of computing integrals over contours encompassing singularities. Additionally, it is a theorem that may be applied to zero-pole qualities and curved integral properties. The Residue Theorem is the basis of many essential mathematical facts revolving around line integrals, particularly in solving ODEs and PDEs and describing physics models.

Keywords: residue theorem, complex analysis, residue.

1. Introduction

1.1. Cardano Formula

Are there only real and positive numbers in the world? This is a question that has plagued mathematicians for a long time. Some examples of this problem were given by early mathematicians in the face of imaginary numbers. Over the centuries, as complex systems such as complex numbers became more widely accepted, so did the study of their contents. Early on, Tartaglia proposed a unique method of solving cubic equations, a breakthrough in the algebraic age, as it began to use abstract reasoning in place of so-called numerical examples. First the Cubic function is express like this:

$$ax^3 + bx^2 + cx + d = 0$$

we get a new function

$$Ax^3 + Cx + D$$

We can get the solution:

$$A = x + \frac{b}{3a}$$

$$C = \frac{3ac - b^2}{3a^2}$$

$$D = \frac{2b^3 - 9abc + 27a^2d}{27a^2}$$

When we have the function

$$x^3 = 15x + 4$$

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$$

So that if we accepted this solution, we must accept the existence of i which means imaginary number.

1.2. Euler formula

Leonhard Euler coined the system of complex numbers - one of the most important and fundamental terms about complex numbers. He said i=$\sqrt{-1}$, which means the square root of a negative number. Euler said that the symbol I is the primary determinant that means the equation cannot really be solved.

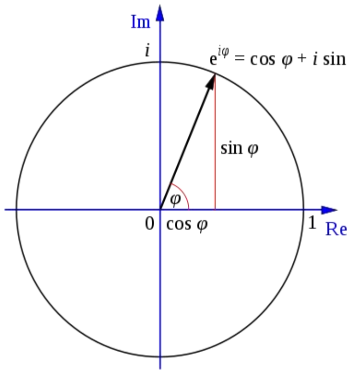

Figure 1. Expression diagram of Euler's theorem.

As is shown in figure 1, one type to the equation of complex analysis:

$$z = x + iy$$

The x is the real part and the latter is the imaginary part, using the Euler formula, we can get the other function type like this:

$$x = rcos\theta$$

$$y = rsin\theta$$

The Euler formula

$$e^{ix} = cosx + isinx$$

So that

$$z = re^{i\theta}$$

2. Cauchy-Goursat therorem

Accoring to the Cauchy-Goursat therorem

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

3. Cauchy integral formula

$$\int_c f(z)dz = 0$$

$f(z)$ can spilt into u+iv, and dz can spilt to dx+idy

$$\int_c u + iv(dx + idy)$$

First we use Green's theorem and then cauchy goursat theorem.

Let $D = \{z \in \mathbb{C}: |z| \leq r\}$ be the closed disk of radius r in $\mathbb{C}$.

Let f:U→$\mathbb{C}$ be holomorphic on some open set containing D.

Then for each a in the interior of D:

$$f(a) = \frac{1}{2\pi i}\oint_{\partial D} \frac{f(z)}{(z-a)}dz$$

where ∂D is the boundary of D, and is traversed anticlockwise.

4. Residue

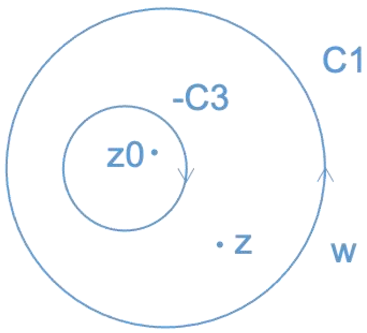

Figure 2. The image describe below.

In figure 2, we have a loop C1 and the clockwise C3 which is negetative.

According to the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \cdot \int_{c1-c3} \frac{f(w)}{w-z}dw$$

$$\int_{c1} \frac{f(w)}{w - z + zo - zo}dw$$

$$= \int_{c1} \frac{f(w)}{(w-z)(1 - \frac{z - z0}{w - zo})}dw$$

$$= \int_{c1} \frac{f(w)}{(w-z)}\sum_{0}^{\infty}(\frac{z-zo}{w-zo})^n dw$$

$$=\sum_{0}^{\infty}(z-zo)^{n}\cdot\int_{c1}\frac{f(w)}{(w-z)^{n+1}}dw$$

$$\int_{c3}\frac{f(w)}{z-zo}\cdot\frac{1}{1-\frac{w-zo}{z-zo}}dw$$

$$=\int_{c3}\frac{f(w)}{(z-zo)}\sum_{0}^{\infty}\frac{w-zo}{z-zo}dw$$

$$=\sum_{-\infty}^{1}(z-zo)^{n}\int_{c3}\frac{f(w)}{(w-zo)^{n}}dw$$

5. Residue Theorem

$u1,u2,\ldots un\subseteq U$(open)

$$a_2\in u2$$

$$a_2\notin uj\ \ i\neq j$$

$$u2\cap uj=\phi$$

Then by the ezistence of Layrent Series

$$f(z)=\sum_{-\infty}^{\infty}a_j(z-z_u)^n$$

$$\int_{\partial u_i}f(z)dz=\int_{\partial u_i}\sum_{-\infty}^{\infty}a_2(z-zo)^n dz$$

$$=\sum_{-\infty}^{-i}\int a_2(z-zo)^n dz+\sum_{0}^{\infty}\int a_2(z-zo)^n dz+\int\frac{a_{-1}}{z-zo}dz$$

$$\int\frac{a_{-1}}{z-zo}dz=2\pi i a_{-1}$$

$$=2\pi i[Res]$$

6. Conclusion

The residue theorem can be used in many ways, that is to say, by obtaining residual formulas for certain functions of several variables. Represent the sum of the values of the polynomial function of a rational convex polyhedron by the Euler-McLaughlin summation formula and integrating [1]. The Residue Theorem can also be used for quadratic refinement by counting the curves on the hypersurface and the complete intersection in the projected space [2]. Also in “Counting Lattice Points by Means of the Residue Theorem” [3], the general residue theorem is used to help derive a new expression such as using the residual theorem to derive an Earhart polynomial. The inner Earhart polynomial and the closure of the tetrahedron are then linked by proving the Earhart-McDonald reciprocity law for the tetrahedron. In “Notes on n-point Witten diagrams in AdS2” [4], the one-dimensional boundary of AdS2 in the Witten diagram allows for a simple setup, and we can also obtain the cause of the error with the residual theorem. In the article “Parametric Euler T-sums of odd harmonic numbers” [5], when we want to establish several formulas for the T-sum of Euler odd harmonic numbers, we need to go through the contour integration method and the remainder theorem.

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Lipstein et al. [6] discovered that using the remainder theorem to obtain a Witten diagram allows to check multiple points of the formula, which provides more details to solve the tree-level single-ring cosmic scattering equation.

The residue theorem also has many implications such as the fact that at zero temperature and a finite chemical potential calculated by integrating over time will yield results consistent with those obtained at small but non-zero temperatures. And applying the remainder theorem to time integration yields a different answer [7]. Simplified analysis in continuous and discrete-time systems may be an alternative to Cauchy's remainder theorem. Simplified methods give analytical results that are more explicit and less restrictive [8]. The method of parentheses (MoB), proposed for evaluating Feynman integrals, gives results in terms of combinations of series(s), and it can also be used to prove the residue theorem [9]. Through A simple evaluation of a theta value and the Kronecker limit formula, we can verify complex functions not only by computing the integral by the residue theorem but also by using elliptic functions or analytical continuations [10].

For example, of the application by Residue Theorem. When account for the anomalous integrals:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$$

then we know

$$z_k = exp(\frac{\pi + 2k\pi}{6} i), k = 0,1,2,3,4,5$$

$$Res(f; z_k) =- \frac{z_k}{6}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} =- 2\pi i \cdot \frac{1}{6} \cdot \sum_{k=0}^{2} z_k = \frac{2\pi}{3}$$

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