Evaluation of limits including integrals by L’ Hôpital’s rule

Xinyao Xu

Shanghai Pinghe School, Shanghai, 201206, China

xuxinyao@shphschool.com

Abstract. Limit is significant concept in mathematic analysis. Technically, limit’s definition in mathematics is that a variable in a function gradually approximates to a certain value in the changing process which cannot be ended. L’ Hôpital’s rule and Taylor expansion, together with other methods such as Stolz theorem, are usually used in measuring a limit’s value. In this paper, it will focus on some representative limits that are related to definite integrals. L’ Hôpital’s rule and Taylor's expansion are also jointed used so as to solve the problems. The main part of this work talks about the limit of the integration of trigonometric function, under which situation Taylor’s expansion is commonly utilized. This article talks about the polynomial’s integration as well, under which situation the approximation method is also employed. Trigonometric function and polynomial function are frequently appeared in evaluating limit. This means that this paper is summarizing the prime functions in integration-related limits.

Keywords: limit, L’ Hôpital’s rule, trigonometric function, Taylor expansion.

1. Introduction

In 1800s, I. Newton and G. W. Leibniz had already found derivatives and integration. However, they didn’t give limit a specific definition which indirectly caused the second crises in Mathematics. In the 19th century, K. Weierstrass gave limit’s definition in the modern sense [1]. Limit’s definition in mathematics is that a variable in a function gradually approximates to a certain value in the changing process which can’t be ended. Limit is imperative to mathematical analysis and calculus. It can help to introduce the definite and indefinite integrals, continuity, and the derivatives of certain functions. Limit plays a critical role in defining whether a function is continuous or not. And the continuity determined the existence of integration and integration of a particular function [2]. When a function’s limit to the left of a value equals to the function’s limit to the right of a value and they are both the same as function’s value at corresponding point, it is said that this function is continuous at this point, which means its derivative is valid nearby. If the whole function is continuous, this specific function is capable to have an indefinite and definite integration. Derivative of a certain function is actually a form of 0/0, which means without the help of limit, it is impossible to get the derivative of that function. Also, the definite integrals are based on dividing the certain parts into some small parts which’s areas approaching to zero and then add up these values. Without limit, the derivates and integrals of certain functions can’t be figure out. Limit is effective in real life as well [3]. For example, with the help of limit, the volume of a conical vase can be valued precisely.

Limit is very useful in many aspects. Limit is usually shown as the ratio of two function such as $\sin(x)/x$. However, in this paper, the authors will talk about the limit which is related to integration

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and this make this work different from others which talk about the conventional limit [4]. By virtue of L' Hôpital's rule, some integration is ready to be done and then limit of the function contained integration can be valued. In what follows, the main part of this article talks about the integration of trigonometric function and the function's limit. The Taylor expansion of trigonometric function is usually used in this kind of situation. This work talks about the polynomial's integration as well. The approximation method is used at that kind of situation.

2. Methods

2.1. L' Hôpital's rule

The L'Hpôital's rule acts as a problem-solving method for deriving the numerator and denominator under certain conditions, and then finding the limit, and finally obtaining the value of the infinitive [5]. According to the current level of cognition, the limit of the ratio of two infinities or the limit of the ratio of two infinities has the possibility of existence, but also the possibility of non-existence. Therefore, when solving the limit of this kind of infinitive, it is necessary to transform it into an important limit form or use the expression form of the limit algorithm to carry out calculations, and the L'Hpôital's rule is a general method for solving such infinitive limits.

Presume that $\lim_{y\to y_0} \phi(y) = \lim_{y\to y_0} \psi(y) = 0$ or $\lim_{y\to y_0} \phi(y) = \lim_{y\to y_0} \psi(y) = \infty$, $\psi'(y) \neq 0$, and $\phi(y)$ is differentiable around y_0, then L'Hpôital's rule states that [6]

$$\lim_{y\to y_0} \frac{\phi(y)}{\psi(y)} = \lim_{y\to y_0} \frac{\phi'(y)}{\psi'(y)} = A. \tag{1}$$

The other types such as $0 \times \infty$ and 1^{∞} can also be calculated via Eq. (1), as they can be changed into the form of $0/0$ and ∞/∞.

Proof. First, this passage will give the proof of L'Hpôital's rule when $\lim_{y\to y_0} \phi(y) = \lim_{y\to y_0} \psi(y) = 0$. Let $\phi(y_0) = \psi(y_0) = 0$, which means $\phi(y)$ and $\psi(y)$ are continuous function in the vicinity of point y_0. The Cauchy mean value theorem is that when $\phi(y)$ and $\psi(y)$ are both continuous at $[c, d]$, are both differentiable at (c, d), $\phi'(y)$ and $\psi'(y)$ are not 0 at the same time, and $\psi(c) \neq \psi(d)$, there is $\varepsilon \in (c, d)$ and then

$$\frac{\phi'(\varepsilon)}{\psi'(\varepsilon)} = \frac{\phi(d) - \phi(c)}{\psi(d) - \psi(c)}. \tag{2}$$

By using Cauchy mean value theorem, it follows that

$$\frac{\phi'(\varepsilon)}{\psi'(\varepsilon)} = \frac{\phi(y) - \phi(y)}{\psi(y) - \psi(y_0)}, \varepsilon \in (y_0, y). \tag{3}$$

Let $y \to y_0$ and $\varepsilon \to y_0$,

$$\lim_{y\to y_0} \frac{\phi(y)}{\psi(y)} = \lim_{\varepsilon\to y_0} \frac{\phi'(\varepsilon)}{\psi'(\varepsilon)} = \lim_{y\to y_0} \frac{\phi'(y)}{\psi'(y)} = Z. \tag{4}$$

This completes the proof. Nevertheless, L'Hpôital's rule can be also used when $\lim_{y\to y_0} \phi(y) = \lim_{y\to y_0} \psi(y) = \infty$, and the procedures of the proof are very similar.

2.2. Taylor's expansion

Taylor's series is a series that uses information about a function at a certain point to describe its value. For any given function that is under some conditions, Taylor's expansion stands out as an extremely useful technique to approximate the function through building a polynomial by using of the values of the derivatives at any given point as coefficients. Taylor's series takes its name from British mathematician Brooke Taylor, who first described it in a letter in 1712. Taylor's series is one of the approximate methods often used to study the properties of complex functions, and it is also an important application of functional differential calculus [7].

The Taylor's expansion is an extremely important tool in dealing with integral, limit, and various approximation methods. Generally, the following expression is amenable to reveal its form:

$$\Xi(y) = \sum_{n=0}^{\infty} \frac{\Xi^{(n)}(y_0)}{n!}(y - y_0)^n. \tag{5}$$

Proof. Any function is able to be expressed in the form $P_n(y) = \sum_{n=0}^{\infty} a_n(y - y_0)^n$ with $a_i\ (i = 1,2,\cdots,n)$ being coefficients. Its first derivative is written as

$$P_n{}'(y) = a_1 + 2a_2y + 3a_3y^2 + 4a_4y^3 + \cdots + na_{n-1}y^{n-1}, \tag{6}$$

and the function's second derivative is written as

$$P_n{}''(y) = 2a_2 + 3 \times 2a_3y + 3 \times 4a_4y^2 + 4 \times 5a_5y^3 + \cdots + (n-1) \times na_{n-2}y^{n-2}. \tag{7}$$

Likewise, the n-th derivative is written as $P_n{}^{(n)}(y) = n!\,a_n$. Taken together, the following equation can be determined: $a_0 = P_n(y_0), a_1 = P_n'(y_0), a_2 = \frac{P_n''(y_0)}{2!}, \dots,\ a_n = \frac{P_n{}^{(n)}(y_0)}{n!}$, as expected by Eq. (5).

3. Representative examples

The first example of the limit is [8]

$$L = \lim_{x\to\infty} \frac{\int_0^x \frac{\sin(g)}{g}dg - \tan(x)}{2x(1-\cos(x))}. \tag{8}$$

To calculate the limit, L'Hopital rule is used here. In this sense, Eq. (8) turns out to be

$$L = \lim_{x\to\infty} \frac{\frac{\sin x}{x} - \frac{1}{\cos^2 x}}{2(1-\cos x + x\sin x)} = \lim_{x\to\infty} \frac{1}{2\cos^2 x} \times \frac{\frac{\sin x}{x}\cos^2 x - 1}{(1-\cos x + x\sin x)} \tag{9}$$

Here, the limit of the first term is a constant of $\frac{1}{2}$ as $\lim_{x\to\infty}\frac{1}{2\cos^2 x} = \frac{1}{2}$, while the second term can be simplified by using Mclaurin polynomial. In light of the formulas regarding $\sin(x)$ and $\cos(x)$, it is observed that

$$L = \frac{1}{2}\lim_{x\to\infty} \frac{\left(1-\frac{1}{6}x^2\right)\left(1-x^2+\frac{1}{4}x^4\right)-1}{\frac{1}{2}x^2+x^2-\frac{1}{6}x^4} = \frac{1}{2}\lim_{x\to\infty} \frac{\left(1-\frac{1}{6}x^2\right)(1-x^2)-1}{\frac{1}{2}x^2+x^2} = -\frac{7}{18}. \tag{10}$$

The second example of the limit is

$$L = \lim_{x\to\infty} \frac{\int_0^x (\arctan t)^2\,dt}{\sqrt{x^2+1}}. \tag{11}$$

Because $\lim_{x\to\infty}\int_0^x (\arctan t)^2\,dt = \infty$ and $\lim_{x\to\infty}\sqrt{x^2+1} = \infty$, L'Hôpital's rule is applicative here. Thus,

$$L = \lim_{x\to\infty} \frac{\frac{d}{dt}\int_0^x (\arctan t)^2\,dt}{\frac{d}{dx}\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{(\arctan t)^2}{\frac{x}{\sqrt{x^2+1}}}. \tag{12}$$

Because $\lim_{x\to\infty}\arctan t = \frac{\pi}{2}$ and $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x} = 1$, so the answer for this function is

$$\lim_{x\to\infty} \frac{(\arctan t)^2}{\frac{x}{\sqrt{x^2+1}}} = \frac{\pi^2}{4}. \tag{13}$$

The third example of the limit is [9]

$$L = \lim_{x\to 0^+} \frac{1}{x^2} \times \int_{\sin x}^{x} (1+\sin u)^{-1/2}\ du. \tag{14}$$

To begin with, the limit can be rewritten as

$$L = \lim_{x\to 0^+} \frac{\int_{\sin x}^{x}(1+\sin u)^{-1/2}\ du}{x^2} = \lim_{x\to 0^+} \frac{\int_0^x (1+\sin u)^{-1/2}\,du - \int_0^{\sin x}(1+\sin u)^{-1/2}\ du}{x^2}. \tag{15}$$

By using of the L'Hôpital's rule, the limit in Eq. (15) turns out to be

$$L = \lim_{x\to 0^+} \frac{(1+\sin x)^{-1/2}\, x\,((x)' - (\sin x)')}{2x} = \lim_{x\to 0^+} \frac{(1+\sin x)^{-1/2} - (1+\sin x)^{-1/2} \times \cos x}{2x}. \quad (16)$$

Therefore, it is arrived that

$$L = \lim_{x\to 0^+} \frac{(1+\sin x)^{-\frac{1}{2}}(1-\cos x)}{2x} = \lim_{x\to 0^+} \frac{1-\cos x}{2x} = \lim_{x\to 0^+} \frac{\sin x}{2} = 0. \quad (17)$$

The fourth example of the limit is

$$I = \lim_{x\to 0^+} \frac{\int_0^{\sin x} \sqrt{\tan t}\ dt}{\int_0^{\tan x} \sqrt{\sin t}\ dt}. \quad (18)$$

In light of L'Hopital's rule, Eq. (18) is equivalent to

$$I = \lim_{x\to 0^+} \frac{\cos x \sqrt{\tan(\sin x)}}{(\sec x)^2\, x\sqrt{\sin(\tan x)}} = \sqrt{\lim_{x\to 0^+} \frac{\tan(\sin x)}{\sin(\tan x)}}. \quad (19)$$

On the other hand, since $\lim_{x\to 0^+} \tan(\sin x)\, x^{-1} = 1$, it is thus

$$I = \lim_{x\to 0+} \frac{\tan(\sin x)}{\sin(\tan x)} = \frac{\lim_{x\to 0+} \tan(\sin x) x^{-1}}{\lim_{x\to 0+} \sin(\tan x) x^{-1}} = 1. \quad (20)$$

The fifth example of the limit is [10]

$$L = \lim_{n\to\infty} \int_0^1 \frac{nx^2+1}{(x^2+1)^n} \log(\cos(n^{-1}x)+2)dx. \quad (21)$$

Let $x = \frac{x}{\sqrt{n}}$, it is easy to find that

$$\int_0^1 \frac{nx^2+1}{(x^2+1)^n} \log\left(2+\cos\left(\frac{x}{n}\right)\right) dx = \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} \frac{x^2+1}{(1+n^{-1}x^2)^n} \log\left(2+\cos\left(\frac{x}{(\sqrt{n})^3}\right)\right) dx. \quad (22)$$

In addition, $\cos\left(\frac{x}{n^{\frac{3}{2}}}\right) \le 1$, $\log\left(2+\cos\left(\frac{x}{n^{\frac{3}{2}}}\right)\right) \le \log 3$ as $n \to \infty$. The orignal expression of hte limit is $L \le \frac{\log 3}{\sqrt{n}} \int_0^\infty \frac{1+x^2}{\left(1+\frac{x^2}{n}\right)^n} dx$. Because $\left(1+\frac{x^2}{n}\right)^n \ge \left(1+\frac{x^2}{2}\right)^2$, the orignal expression is smaller than

$$\frac{\log 3}{\sqrt{n}} \int_0^\infty \frac{1+x^2}{\left(1+\frac{x^2}{2}\right)^2} dx = \frac{\log 3}{\sqrt{n}} \int_0^\infty \frac{4+4x^2}{4+x^4+4x^2} dx = \frac{\log(3)}{\sqrt{n}} \left(\frac{3\sqrt{2}\pi}{4}\right), \quad (23)$$

which means that the limit of L tends to zero as $n \to \infty$.

4. Conclusion

Limit is significant in mathematic analysis. By definition, limit in mathematics is that a variable in a given function gradually approximates to a certain value in the changing process which can't be ended. A limit's value is typically determined using the Taylor expansion and L'Hôpital's rule. This paper has emphasized the limit of integration through several representative examples. This work also makes use of Taylor's extension and L'Hôpital's rule to solve them thoroughly. Notably, majority of this work is concerned with the limit of trigonometric function integration, a scenario in which Taylor's expansion is frequently employed. Specifically, questions 1, 2, 3, and 4 are about the integration of trigonometric function and the limit of it. This article also discusses the polynomial's integration in question 5, a condition in which the approximation method is frequently employed. In evaluating limits, polynomial and trigonometric functions emerge frequently. This means that the prime functions in integration that are connected to the limit are summarized in this essay. Before solving these questions, this article also gives the L' Hôpital's rule's proof and the definition of Taylor's expansion. To summarize, this work provides a comprehensive demonstration of the vital role of the limit in calculus.

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