DOI: 10.54254/2753-8818/10/20230321

Approaches to solving several definite integrals with special functions

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Abstract. Calculus is the foundation of many natural sciences such as Physics. Calculus excels at calculating the area of irregularly shaped objects and thus it may be used in a vast array of domains. Because calculus is difficult to perform when combined with trigonometric and logarithmic functions, additional formulas are required to assist with calculations. By definition, limit of the sum of a function $f(x)$ across given interval $[a, b]$ is the definite integral. Notice the relationship between definite and indefinite integrals is as follow: result of a definite integral is a precise nice value, whereas an indefinite integral is expressed by a function. Their mathematical relationship is limited to computation regarding the Newton-Leibniz formula. This article describes only one of several methods for calculating definite integrals. Taylor expansion will also be used for auxiliary operations, while the relevant equations of Taylor expansion will also be presented in the text. It will also be learned through this paper that the result of the integral varies with π.

Keywords: definite integrals, improper integrals, calculus, special functions.

1. Introduction

Calculus, formerly known as infinitesimal calculus, is a branch of mathematics concerned with limits, continuity, derivatives, integrals, and infinite series. Calculus is the most fundamental subject in mathematics and has its origins in ancient Rome. Several theories, such as those related to Einstein, use calculus to aid in their verification [1]. In the 17th century, nearly all of the masters of science devoted themselves to addressing the issues of rates, extrema, tangents, areas, and most notably, the infinitesimal algorithm describing motion and variation, and achieved rapid progress. Astronomer Kepler discovered three laws of planetary motion and used the concepts of infinitesimal sums to calculate the areas of edges of curves and the volumes of rotating objects. Cavalier discovered Cavalieri's principle (zuccio's principle) and Girtin's theorem using the indivisible measure and the formula for definite integrals of power functions at about the same time. Cavalieri also proved Gilding's theorem (the volume of a three-dimensional figure obtained by rotating a 2D figure around an axis is the same as the product of the circumference of the circle). This had a significant impact on the early development of calculus. In addition, the algebraic methods of the French mathematician Descartes, the father of analytic geometry, contributed significantly to the growth of calculus [2]. The eminent French mathematician Fermat made significant contributions to the determination of curve tangents and function extrema. Among them is Fermat's theorem on mathematical analysis: "Let the function $f(x)$ be defined in some interval, and take

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its greatest (minimum) value at the innermost point c." If a finite derivative $f'(c)$ exists at this point, then $f'(c)$ must equal 0.

Inspired by Wallis's Infinity Arithmetic, the English scientist Newton pioneered the study of calculus and for the first-time extended algebra to analysis. Newton introduced positive stream numbering (differentiation) in 1665, followed by inverse stream numbering in 1666. Thereafter, he summarized flow mathematics and published "A Short Explanation of Flow Mathematics," which marked the start of calculus. In the later stages of Newton's calculus, he rejected his earlier view that the variable is a static collection of infinitesimally small elements and no longer emphasized that the mathematical quantity is composed of the smallest indivisible unit, believing instead that it is generated by geometric elements in continuous motion. This is a shift from the original perspective of real, infinitesimal quantities to the perspective of the infinite division of the quantities, which represents the perspective of potential infinity [3].

Calculus is the most fundamental mathematical discipline. Calculus demonstrates numerous hypotheses and promotes the development of mathematics. With calculus, mathematics can describe the motion of objects and the evolution of a process. Calculus is the cornerstone of physical theory, and Newton derived the three physical laws using differential equations. There are numerous ways for calculating the definite integral, such as variable substitution, the series approach, the Fourier transform, and the Laplace transform [4]. Typically, the definite integral is used to compute the area of irregular figures.

2. Integration techniques and Taylor expansions

2.1. Integration techniques

There are many methods to calculate integral. The divisional integration formula is perhaps the simplest one. It states that $(uv)' = u'v + uv'$ and thus $u'v = (uv)' - uv'$. The points are scored simultaneously by both terms, namely, $\int u'vdx = \int (uv)'dx - \int uv'dx$. Consequently, $\int u'vdx = uv - \int uv'dx$. Moreover, there are also many other methods. The integration of trigonometric functions deals with integrals involving trigonometric functions [5]. It focuses specifically on the deformations of cos and sin, as well as the transformation from logarithms to trigonometric functions.

The trigonometric substitution deals with the transformation relationships between trigonometric functions in order to simplify a number of integrals. The approximation of definite integrals states that, while it is not always possible to obtain exact values, this method will at least enable people to obtain an approximation, which is often sufficient.

2.2. Taylor expansions

The Taylor's formula specifies a function that takes values nearby by virtue of data at a particular place. As long as the function satisfies certain conditions, Taylor's formula permits the building of a polynomial that approximates the expression of this function by employing the values of derivatives as coefficients. Taylor's formula is a fundamental component of mathematical analysis; it is an invaluable tool for examining the limits of functions and assessing mistakes [6]. Taylor's expansion can convert non-linear issues into linear with great precision, making it essential to all areas of calculus. Taylor's formula has the geometric sense of approximating the original function with a polynomial function. Since polynomial functions can be derived arbitrarily, are simple to calculate, and can be used to solve for extreme values or determine the nature of the function, the Taylor formula can be used to obtain information about the function; whereas an error analysis must be provided for this approximation to ensure its reliability.

There are several famous expansions of known functions. For example, it is found that [7]

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, -\infty < x < \infty, \tag{1}$$

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$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, -1 < x < 1, \tag{2}$$

$$\arctan x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots, -1 < x < 1, \tag{3}$$

$$\tan x = 1 + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots, -1 < x < 1, \tag{4}$$

and

$$\zeta(\mathrm{x}) = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{3^x} + \cdots, x > 1. \tag{5}$$

3. Examples and applications

This section seeks to solve multiple integrals with inverse tangent function and logarithmic function.

3.1. Example I

The first representative integral is [8]

$$I = \int_0^1 \frac{\tan^{-1} x}{x} \ln\left(\frac{1+x^2}{(1-x)^2}\right) = \frac{\pi^3}{16}. \tag{6}$$

The first step to solve Eq. (6) is by virtue of integrand by parts. It follows that

$$I = \int_0^1 \frac{\ln x}{1+x^2} \ln \frac{1+x^2}{(1-x)^2} dx - 2\int_0^1 \frac{(1+x)\ln x \tan^{-1} x}{(1-x)(1+x^2)} dx\,. \tag{7}$$

Before proceeding further, it is useful to introduce an auxiliary function

$$R(\mathrm{x}) = \int_0^x \frac{(1+t)\ln t}{(1-t)(1+t^2)} dt = \int_0^1 \frac{x(1+tx)\ln tx}{(1-tx)(1+t^2x^2)} dt \tag{8}$$

where $x \in [0,1]$. Obviously, $R(1) = \int_0^1 \frac{t\ln t}{1+t} dt + \int_0^1 \frac{\ln t}{1-t} dt$. By using integrand by parts again, it is found that $I = I_1 + I_2$, where

$$I_1 = -\frac{\pi}{2} R(1) - \frac{\pi}{2}\int_0^1 \frac{t\ln t}{1+t^2} dt - \frac{\pi}{2}\int_0^1 \frac{t\ln t}{1-t^2} dt \tag{9}$$

and

$$I_2 = \ln 2 \int_0^1 \frac{\ln t}{1+t^2} dt - 2\int_0^1 \frac{\ln(1-t)\ln t}{1+t^2} dt + 2\int_0^1 \frac{\ln t \arctan t}{1-t^2} dt\,. \tag{10}$$

Likewise, introducing another function

$$S(x) = \int_0^x \frac{\ln t}{1-t^2} dt = \int_0^x \frac{x\ln(tx)}{1-t^2x^2} dt \tag{11}$$

and $S(1) = \int_0^1 \frac{\ln t}{1-t^2} dt$. In light of Eq. (11), it is found that

$$\int_0^1 \frac{\ln x \arctan x}{1-x^2} dx = \frac{\pi}{4} S(1) - \frac{\ln 2}{2}\int_0^1 \frac{\ln t}{1+t^2} dt + \int_0^1 \frac{\ln(1-x)\ln x}{1+x^2} dx\,. \tag{12}$$

Thus, the total integral is

$$I = \pi \int_0^1 \frac{2t\ln t}{t^4-1} dt = \frac{1}{2}\pi \int_0^1 \frac{\ln y}{y^2-1} dy = \frac{\pi^3}{16}. \tag{13}$$

where the substitution $y = t^2$ is used.

3.2. Example II

The second representative integral is [9]

$$I = \int_0^1 \frac{\tan^{-1} x}{x} \ln\left(\frac{1+x}{\sqrt{1+x^2}}\right) dx. \tag{14}$$

The first step to solve Eq. (14) is by virtue of integrand by parts. It follows that

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$$\frac{1}{3}\int_0^1 \frac{\tan^{-1} x \ln(1+x)}{x} dx = \frac{1}{2}\int_0^1 \frac{\tan^{-1} x \ln(1+x^2)}{x} dx. \tag{15}$$

The transformation according to Eq. (15) results that

$$I = \int_0^1 \frac{\tan^{-1} x}{x} \ln\left(\frac{1+x}{\sqrt{1+x^2}}\right) dx = \frac{2}{3}\int_0^1 \frac{\tan^{-1} x \ln(1+x)}{x} dx. \tag{16}$$

Let Eq. (16) be in the form of a Riemann sum, then it can get

$$I = -2\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}}{2n+1} \int_0^1 x^{2n}\, dx = -2\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}}{(2n+1)^2} \tag{17}$$

After simplifying the above equation, it is readily to get

$$I = -2\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{(2n+1)^2} + 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = -2\Im \sum_{n=1}^{\infty} \frac{i^n H_n}{n^2} + \frac{\pi^3}{16}. \tag{18}$$

Using the generating function with $x = i$,

$$\Im \sum_{n=0}^{\infty} \frac{i^n H_n}{n^2} = -\frac{\pi}{16}\ln^2 2 - \frac{1}{2} G \ln 2 - \Im Li_3(1-i) \tag{19}$$

where the celebrated Catalan's constant G is involved. After substituting the result of Eq. (19) into Eq. (18), one gets

$$I = 2\xi\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) + G\ln 2 - \frac{3}{64}\pi^3 - \pi\frac{1}{16}\ln^2 2. \tag{20}$$

3.3. Example III

The third representative integral is [10]

$$\mathrm{I} = \int_0^1 \frac{(\tan^{-1} x)^2 \ln \frac{x}{(1-x)^2}}{x} dx \tag{21}$$

The first step to solve Eq. (21) is by virtue of integrand by parts. It follows that

$$\boldsymbol{I = \int_0^1 \frac{\left(\tan^{-1} x\right)^2}{1+x^2} \ln\left(1+x^2\right) dx = -2\int_0^{\frac{\pi}{4}} u^2 \ln(\cos(u)) du}. \tag{22}$$

Before proceeding further, it is useful to employ the Fourier series

$$\ln\left(\cos(u)\right) = -\ln(2) - \sum_{k\geq 1} \frac{(-1)^k \cos(2ku)}{k}, 0 \leq x \leq \frac{\pi}{2}. \tag{23}$$

When substituting Eq. (23) into Eq. (22), it follows that

$$I = \frac{\ln(2)\pi^3}{96} + 2\sum_{k\geq 1} \frac{(-1)^k}{k} \int_0^{\frac{\pi}{4}} u^2 \cos(2ku) du. \tag{24}$$

Before calculating, by performing a partial simplification, it is arrived that

$$\int_0^{\frac{\pi}{4}} u^2 \cos(2ku) du = \frac{\pi^2 \sin\frac{\pi}{2}k}{32k} - \frac{\sin\frac{\pi}{2}k}{4k^3} + \frac{\pi\cos\frac{\pi}{2}k}{8k^2}. \tag{25}$$

Substitute Eq. (25) into Eq. (24), it is obtained that

$$I = \frac{\ln(2)\pi^3}{96} + \pi^2\sum_{k\geq 1} \frac{(-1)^k \sin\frac{\pi}{2}k}{16k^2} - \sum_{k\geq 1} \frac{(-1)^k \sin\frac{\pi}{2}k}{2k^4} + \pi\sum_{k\geq 1} \frac{(-1)^k \cos\frac{\pi}{2}k}{4k^3}. \tag{26}$$

The approximate range is

$$\cos\left(\frac{\pi k}{2}\right) = \begin{cases} -1, k \equiv 2 \bmod 4 \\ 1,\ k \equiv 0 \bmod 4, \\ 0 \quad \text{otherwise} \end{cases} \qquad \sin\left(\frac{\pi k}{2}\right) = \begin{cases} -1, k \equiv 3 \bmod 4 \\ 1,\ k \equiv 1 \bmod 4. \\ 0 \quad \text{otherwise} \end{cases} \tag{27}$$

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By simplifying above, it is found that that

$$I = \frac{\ln(2)\pi^3}{96} - \frac{\pi^2}{16}K + \frac{\beta(4)}{2} - \frac{3\pi\zeta(3)}{128} \approx 0.064824. \tag{27}$$

4. Conclusion

There are many formulas used to assist in the calculation of definite integrals. Taylor's formula is one of the more commonly used formulas. Most people usually use the result of the expansion of Taylor's formula to perform operations and the details of Taylor's formula are explained in detail in the paper. The method exemplified in the paper can make the calculation of definite integrals faster and more accurate. The divisional integral formula is also used in the paper. The divisional integral formula is also a relatively common knowledge point somewhat similar to the substitution method. An equation is substituted for a letter and replaced with the relevant equation from the letter change. At the end, the substituted equation is used to perform definite integral operations. Taylor's expansion and the division integral formula are the basis for calculating definite integrals. Most of the rest of the equations need to use these two equations to build the foundation.

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