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On Euclidean, spherical and hyperbolic crystallography

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Abstract. A major impetus in the early development of group theory in the 19th century was the study of geometrical symmetries. Inspired by the theory regarding orbifold notation developed by John Conway, we understand and analyze the theory of the classification of symmetrical patterns in three different spaces. One of the triumphs was the full understanding of 2-dimensional planar symmetries, precisely, the classification theorem of wallpaper groups. Then we use the same way to classify the symmetrical patterns in spherical and hyperbolic spaces. There were also scattered theories on the general notion called crystallographic groups. In this paper, we reproduce the classical result that there are 17 types of wallpaper groups using a topological method; we also conclude that there are 14 family cases in the spherical space and infinite conditions in the hyperbolic spaces. During this process, we first consider the three different spaces, then, analyze the symmetrical patterns case by case. The characteristic we consider to classify these patterns is orbifold; finally, we use the formulas to calculate the result.

Keywords: Symmetrical patterns, isometry groups, orbifold notation.

1. Introduction

The symmetry is an important concept for both mathematical studies and human aesthetics. Symmetrical patterns are frequently exhibited in paintings and designs. Here are some examples in ancient times (pictures taken from [1]):

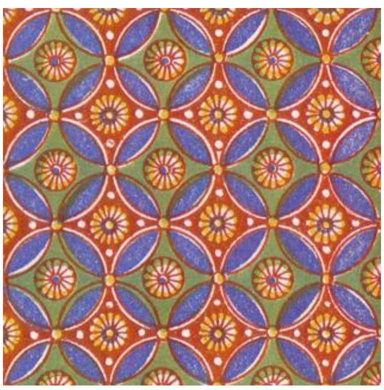

Figure 1. An Egyptian mummy case.

Figure 2. A Chinese painting.

Figure 3. A Persian ornament.

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Here are some quotidian examples (picture taken from [2]):

Figure 4. Tennis ball, basketball, soccer ball and volleyball [2].

Some symmetrical patterns, though visually look different, are actually "the same" in an abstract point of view. See for example the two objects below (pictures taken from [2]):

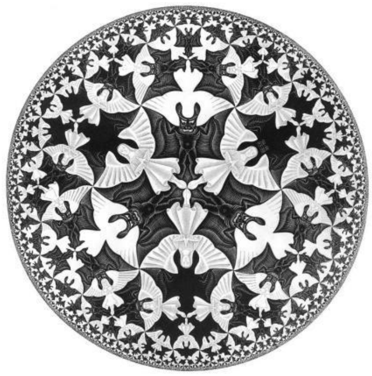

Figure 5. M. C. Escher, Angles and Devils [2].

The distorted icons here are in fact manifestations of some exotic symmetry. This intriguing piece of artwork suggests that there might be some rich mathematical theory behind it, which we will explore.

2. Crystallographic groups

2.1. Isometries

Let X denote a metric space throughout. The main examples are the Euclidean space $\mathbf{E}^n$; the sphere $\mathbf{S}^n$, viewed as a geodesic Riemannian manifold; the hyperbolic space $\mathbf{H}^n$. The notion of symmetries of X is formulated as follows.

Definition 2.1. A symmetry / isometry / isometric map of X is a bijection $f: X \to X$ that satisfies

$$\mathrm{d}\big(\mathrm{f}(\mathrm{x}), \mathrm{f}(\mathrm{y})\big) = \mathrm{d}(\mathrm{x}, \mathrm{y}),\ \forall \mathrm{x}, \mathrm{y} \in \mathrm{X}.$$

All isometries of X under compositions form a group, called the symmetry group / isometry group of X, denoted as $\mathrm{Iso}(X)$.

Example. Let X be a cube in $\mathbf{E}^3$. It has the following symmetries:

- Rotation along a line passing a pair of opposite face centers by $\frac{k\pi}{2}$ $(k = 1,2,3)$. There are 9 of them.
- Rotation along a line passing a pair of opposite vertices by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. There are 8 of them.
- Rotation along a line passing a pair of midpoints of opposite edges by π. There are 6 of them.

- Reflection along some plane. The number of reflections is the same as the number of rotations (plus the identity), which is 24.

These symmetries form a group of order 48 called the full octahedral group, denoted as O'. The first three types (those are, orientation-preserving isometries) form an index 2 subgroup O, called the octahedral group. It has the presentation

$$O = \langle a, b, c | a^4 = b^3 = c^2 = abc = id \rangle$$

where a, b, c are of the 1st, 2nd and 3rd type respectively. And

$$O' = O \times \mathbf{Z}/2.$$

The isometry group Iso(X) acts naturally on X, hence induces an action of any subgroup G ⊂ Iso(X) on X.

Definition 2.2. A singularity of G is a point $x \in X$, such that the stabilizer of x is non trivial.

Definition 2.3. A fundamental domain of G is an subspace $Y \subset X$, such that for any G-orbit $\mathcal{O}$, $|\mathcal{O} \cap Y| = 1$.

Remark. If Y is a fundamental domain, then so is $g \cdot Y$ for any $g \in G$. In practice, we often choose and fix a fundamental domain Y (though the choice is highly non canonical), then say "a fundamental domain" to refer to $g \cdot Y$ for some $g \in G$.

Definition 2.4. The orbifold of G is the space X/G, endowed with the quotient topology.

Intuitively, the orbifold is obtained from curling a fundamental domain by possibly identifying some pieces of its boundary, as illustrated by the example below.

Example. Let $X = \mathbf{S}^2$. A rotation by $\frac{2\pi}{n}$ generates a cyclic subgroup of order n of $\mathrm{Iso}(\mathbf{S}^2)$. A corresponding fundamental domain can be taken as a spherical lune of dihedral angle $\frac{2\pi}{n}$. And by identifying the two great arcs one obtains the orbifold, which is topologically a sphere again.

Definition 2.5. We say that G is cocompact, if the orbifold X/G is compact.

In particular, if X is itself compact (e.g. $X = \mathbf{S}^n$), then every subgroup of $\mathrm{Iso}(X)$ is cocompact since the continuous image of a compact space is compact.

Definition 2.6. We say that G is discrete, if for any G-orbit O and any compact subspace $Y \subset X$, $|\mathcal{O} \cap Y| < \infty$.[3]

Remark. When X is complete and locally compact (e.g. X is a manifold), the discrete condition means that no G-orbit admits an accumulation point. **Proposition 2.7**. If X is a geodesic manifold, G is discrete and $x \in X$ is a singularity, then either x is an isolated singularity, or the set of singularities contains a geodesic passing x.

Proof: Suppose x is not isolated, then there are other singularities x_i ($i \in \mathbf{N}$) such that $x_i \to x$. Hence for each i sufficiently large, there is a unique geodesic l_i connecting x_i and x. For each such i, take $f_i \in G - \{id\}$ such that $f_i(x_i) = x_i$. Then $d(x, f_i(x)) \leq d(x_i, x) + d(x_i, f_i(x)) = 2d(x_i, x) \to 0$ as $i \to \infty$. But the orbit of x can not have accumulation point, so there exists some i such that $f_i(x) = x$. It follows that f_i fixes l_i.

Remark. For $X = \mathbf{E}^2, \mathbf{S}^2, \mathbf{H}^2$, if f_i fixes l_i, then f_i is the reflection along l_i.

Definition 2.8. We say that G is crystallographic, if it is cocompact and discrete.

Proposition 2.9. If G is crystallographic and H is a finite index subgroup of G, then H is also crystallographic.

Proof: It follows immediately from the definition that every subgroup of G is discrete. The natural map $X/H \to X/G$ is finite (the preimage of any point has at most $[G: H]$ points), hence X/G compact implies that X/H is compact. [4]

2.2. Symmetrical patterns

n the various examples we have seen, we observe that the patterns are symmetrical, in the sense that there is an icon which keeps repeating itself all over the space. Put it mathematically, an icon is sent to another identical one by some isometry, and all such icons form a partition (disjoint union) of the space. Thus, if we take all isometries that preserve the pattern, they form a subgroup G of the whole isometry group, under whose group action an icon is a fundamental domain. For the pattern to be well visualized, one requires that an icon has an area which is neither zero nor infinity. The first requirement (nonzero) means that G should be discrete, and the second one means that G should be cocompact. Thus, we are essentially studying this symmetry group G behind the pattern, and the assumption we need to impose on G is that it is a crystallographic subgroup. The upshot is: two symmetrical patterns are the same if their symmetry groups are isomorphic, and the problem of classifying symmetrical patterns is translated as the mathematical problem of classifying crystallographic subgroups up to isomorphisms.

Consider the sphere $\mathbf{S}^n = \{x \in \mathbf{E}^{n+1} | \langle x, x \rangle = 1\}$. We pin down the problem more concisely. The isometry group of $\mathbf{S}^n$ is described in linear algebra as the orthogonal group

$$\mathrm{Iso}(\mathbf{S}^n) = O(n+1) = \{A \in M(n+1; \mathbf{R}) | AA^T = id\}.$$

That is, the isometries are linear automorphisms of $\mathbf{E}^{n+1}$ that preserve the inner product. It has an index 2 subgroup $SO(n+1)$ consisting of orientation-preserving (i.e. $det(A) = 1$) maps.

Example. For $n = 1$, the orientation-preserving isometries of $\mathbf{S}^1$ are rotations, so $SO(2) \cong S^1$. The orientation-reversing isometries are reflections along a diameter.

Proposition 2.10. The crystallographic subgroups of $\mathrm{Iso}(\mathbf{S}^n)$ are exactly its finite subgroups.[3]

3. The spherical case

3.1. *The circle patterns*

We ask for finite subgroups of $O(2)$. Examples arise as follows: take a regular n-gon X, it has a circumscribed circle $\mathbf{S}^1$. The isometry group D_n of X then acts faithfully on $\mathbf{S}^1$, thus is embedded into $\mathrm{Iso}(\mathbf{S}^1)$. Note that the embedding is only unique up to conjugation, because $\mathbf{S}^1$ has many inscribed regular n-gons, all differ by a rotation from each other. Thus D_n is a finite subgroup of $O(2)$, so is its subgroup $D_n \cap SO(2)$, which is isomorphic to $C_n := \mathbf{Z}/n$.

Lemma 3.1. Finite subgroups of S^1 are C_n ($n \in \mathbf{N}^*$).

Proof: As a set $S^1 = (0, 2\pi]$. If $H \subset S^1$ is finite, there exists a minimal $\theta \in H$. Suppose θ is of order n, then $H = \{k\theta | k = 1, \cdots, n\}$. Otherwise, if $\gamma \in H$ is not an integer multiple of θ, then γ mod θ is in H, which contradicts the minimality of θ. So $H \cong C_n$.

Proposition 3.2. If $G \subset O(2)$ is finite, then $G \cong D_n$ or C_n for some $n \in \mathbf{N}^*$.

Proof: The above lemma says that $G \cap SO(2)$, as a finite subgroup of $SO(2) = S^1$, is cyclic. By lemma 2.15, this cyclic group C_n is either G itself, or an index 2 subgroup of G. In the latter case, G contains a reflection r, and is generated by C_n and r. So $G \cong D_n$.

We can describe the corresponding orbifold types.

- For $G = C_n$, the orbifold $\mathbf{S}^1/C_n$ is a topological circle again. The action is free, hence $\mathbf{S}^1 \to \mathbf{S}^1/C_n = \mathbf{S}^1$ is the degree n covering map.
- For $G = D_n$, the orbifold $\mathbf{S}^1/D_n$ is homeomorphic to $[0,1]$. The action has $2n$ singularities, half are mapped to 0 by $\mathbf{S}^1 \to \mathbf{S}^1/D_n$ and half are mapped to 1. Apart from the singularities, the map is a covering of degree $2n$.

3.2. *Finite subgroups of* $O(3)$

We now turn to the case $n = 2$. We first want to understand what are the elements in $O(3)$.

Proposition 3.3. Every isometry of $\mathbf{S}^2$ is either a rotation along a diameter, or a reflection along a great circle, or a glide reflection (the composition of the previous two).

Example. The antipodal map $\tau := -id$. It is a glide reflection (reflection plus a rotation by π). It is the only non trivial element in $O(3)$ that commutes with all elements.

We give two proofs, first a geometrical one then an algebraic one.

Proof: Let $f \in \mathrm{Iso}(\mathbf{S}^2)$. We consider two cases.

- If f has a fixed point x, then f preserves any circle centered at x, and is determined by its action on one such circle (in fact f is already determined by its value on three non collinear points). By considering this restricted action the case is reduced to the classification of $\mathbf{S}^1$-isometries, which we already know are rotations and reflections. So f is a $\mathbf{S}^2$ rotation or reflection.
- If f is fixed point free. We employ the same trick that is to compose a reflection with f to force a fixed point hence reduce to the first case. Note that two reflections must make a rotation (because on $\mathbf{S}^2$ any two great circles intersect), so f can only be a composition of rotation and reflection, which is a glide reflection.

Proof: Let $A \in O(3)$. The characteristic polynomial of A is cubic, hence must admit a real root λ. Hence $\mathbf{R}^3$ has a 1-dimensional invariant subspace $\mathbf{R}v$. All the roots are of modulus 1 since A is orthogonal, so $\lambda = \pm 1$.

1. $Av = v$, then A is the direct sum of 1 and some $B \in O(2)$. Such matrix represents a rotation or a reflection.
2. $Av = -v$, then A is the direct sum of -1 and some $B \in O(2)$. Such matrix represents a glide reflection or a rotation by π. Alternatively, by composing τ and A we are reduced to the first case. The composition of τ with a rotation is still a glide reflection. That with a reflection is a rotation by π.

Next, we want to have some examples of finite subgroup of $O(3)$. Since $O(2)$ can be embedded in $O(3)$, all finite subgroups of $O(2)$ are also finite subgroups of $O(3)$. Furthermore, inspired by the examples of $O(2)$ arising from inscribed regular gons, we feel that the problem is related to the inscribed regular polyhedra of $\mathbf{S}^2$. There are only five of them, which has been known since ancient Greece.

Theorem 3.5. Up to conjugation, finite subgroups of $SO(3)$ are

1. The cyclic group C_n of order n, embedded via $C_n \subset SO(2) \subset SO(3)$.
2. The dihedral group D_n of order $2n$, embedded via $D_n \subset O(2) \subset SO(3)$.
3. The tetrahedral group T, consisting of rotation symmetries of tetrahedron. It has order 12 and the presentation[5]
$$T = \langle a, b, c | a^3 = b^3 = c^2 = abc = id \rangle.$$
4. The octahedral group O, consisting of rotation symmetries of octahedron. It has order 24 and the presentation
$$O = \langle a, b, c | a^4 = b^3 = c^2 = abc = id \rangle.$$
5. The icosahedral group I, consisting of rotation symmetries of icosahedron. It has order 60 and the presentation
$$I = \langle a, b, c | a^5 = b^3 = c^2 = abc = id \rangle.$$

Remark. One can show that $T \cong A_4$, $O \cong S_4$ and $I \cong A_5$. Moreover, T is a subgroup of O, because there are four vertices in the cube that form a tetrahedron.

To pass from the above to the result concerning $O(3)$, we use the coset decomposition $O(3) = SO(3) \cup \tau SO(3)$. If G is a finite subgroup of $O(3)$ but not of $SO(3)$, then $H := G \cap SO(3)$ is one of the above, and $G = H \cup \tau a H$ for some $a \in SO(3)$. The fact that G is a group implies that $a^2 \in H$ and $aH = Ha$. It holds automatically if $a \in H$, then $G = H \cup \tau H = H \times C_2$, which gives us five more subgroups:

1. $C_n' := C_n \times C_2$ of order $2n$.
2. $D_n' := D_n \times C_2$ of order $4n$.
3. $T' := T \times C_2$ of order 24.
4. $O' := O \times C_2$ of order 48, consisting of all symmetries of octahedron.
5. $I' = I \times C_2$ of order 120, consisting of all symmetries of icosahedron.

If $a \notin H$, then $K := H \cup aH$ is a subgroup of $SO(3)$ such that $H \subset K$ is of index 2. Conversely, given any such pair (H, K), there is a subgroup of $O(3)$ defined by

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$$K * H := H \cup \tau(K - H).$$

Searching for the pairs (H, K) in the above theorem, we get four more subgroups:

1. $O * T$ of order 24, consisting of all symmetries of tetrahedron.
2. $C_{2n} * C_n$ of order $2n$.
3. $D_n * C_n$ of order $2n$.
4. $D_{2n} * D_n$ of order $4n$.

Theorem 3.6. Up to conjugation, every finite subgroup of $O(3)$ belongs to one of the 14 types mentioned above.

We prove this result in the next section. Our proof is topological in essence. Specifically, the orbifold type suffices to classify the groups. The below graphic presentation of the group actions (following Conway's orbifold notation in [2], to be explained soon) helps to visualize the groups.

- The groups C_N, D_N, T, O, I correspond to Conway's notation NN, $22N$, 332, 432, 532:

Figure 6. Five more Conway's notations with the corresponding group in bracket: $*NN$ $(D_N * C_N)$, $*22N$ (D_N' when $2 \mid N$, $D_{2N} * D_N$ when $2 \nmid N$), $*332$ $(O * T)$, $*432$ (O'), $*532$ (I').

Figure 7. Four hybrid types: $3{*}2$ (T'), $N*$ (C_N' when $2 \mid N$, $C_{2N} * C_N$ when $2 \nmid N$), $2{*}N$ ($D_{2N} * D_N$ when $2 \mid N$, D_N' when $2 \nmid N$), $N\times$ ($C_{2N} * C_N$ when $2 \mid N$, and C_N' when $2 \nmid N$).

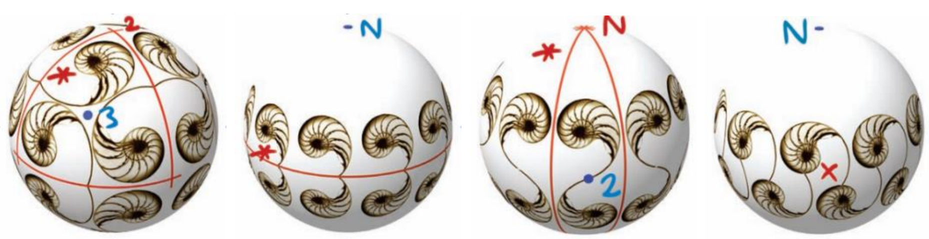

Figure 8. Example. In figure 4, the balls have symmetry $2{*}2$, $*222$, $*532$, $3{*}2$ respectively. In figure 5, both have symmetry 532.

3.3. The orbifold classification

Let $G \subset O(3)$ be a finite subgroup. Then the orbifold $\mathbf{S}^2/G$ is a compact surface. The following classification theorem is well-known:

Theorem 3.7. Every closed (i.e. compact and boundaryless) surface S is homeomorphic to:

1. When S is orientable, a sphere adding n handles, denoted as nT^2 ($n \in \mathbb{N}$). The Euler number $e(nT^2) = 2 - 2n$.
2. When S is non-orientable, a sphere adding m crosscaps, denoted as mP^2 ($m \in \mathbb{N}^*$). The Euler number $e(mP^2) = 2 - m$.

Every compact surface is homeomorphic to a closed surface S punched k times, for some S and $k \in \mathbf{N}$, denoted as S_{-k}. The Euler number $e(S_{-k}) = e(S) - k$.

Thus, to determine the homeomorphism type of a compact surface M, there are three things to do:

1. Determine k. Since ∂M is the disjoint union of k circles, we simply count the number of connected components of ∂M.
2. Determine whether M is orientable or not.
3. Determine S. To this end, it suffices to compute the Euler number $e(M)$.

To classify the orbifold $\mathbf{S}^2/G$ is not merely to classify the underlying topological surface, but also we need to take the singularities into account. Singularities occur whenever some element in G fixes a point. Recall that such element is either a rotation (fixing a pair of antipodal points) or a reflection (fixing a great circle).

Definition 3.8. We say that a singularity $x \in X$ is a cone point if $\mathrm{Stab}(x)$ consists of rotations.

Since $\mathrm{Stab}(x)$ acts faithfully on a small circle centered at x, it embeds as a finite subgroup of $O(2)$. Now we have proposition 3.2 ready.

Definition 3.9. A cone point x is said to have order n, if $\mathrm{Stab}(x)$ is cyclic of order n.

So locally at a cone point of order n, the orbifold is $\mathbf{D}^2/C_n$.

Definition 3.10. We say that a singularity $x \in X$ is a corner point of order n, if $\mathrm{Stab}(x) \cong D_n$.

Remark. The orbifold is $\mathbf{D}^2/D_n$ locally at a corner point of order n. From now on we only use "corner point" to refer to that of order ≥ 2, and that of order 1 is referred to as "boundary point", for the orbifold is a closed half plane locally.

Apart from the singularities the natural projection $p: \mathbf{S}^2 \to \mathbf{S}^2/G$ is a $|G|$-sheeted covering, because the restricted action is free. This is good because of the following.

Lemma 3.11. For any n-sheeted covering $p: X \to Y$ of compact manifolds, $e(X) = ne(Y)$.

Proof: Take any triangulation of Y, refine it so that every simplex is contained in a trivializing open subset (i.e. open subset U such that $p^{-1}(U) \cong \coprod_n U$). Then the refined triangulation can be lifted along p, so that the number of simplexes of X in each dimension is n times the number of Y. The result then follows.

Informally, the lemma implies that $e(\mathbf{S}^2/G)$ is approximately $\frac{e(\mathbf{S}^2)}{n} = \frac{2}{n}$, and the error is controlled by the singularities. So the singularities not only form part of the data needed in the orbifold classification, but also are involved in the formula of $e(\mathbf{S}^2/G)$. Now we derive this explicit formula.

Denote by F the set of singularities. It consists of all axes of reflections in G, and all centers of rotations in G. Let $A \subset F$ denote the union of those axes, and $B := F - A$ which consists of some isolated points. Some simple observations:

1. $p(A) = \partial(\mathbf{S}^2/G)$. Since all great circles on $\mathbf{S}^2$ intersect, A is path connected. So $\mathbf{S}^2/G$ has at most one piece of boundary.
2. B is the set of cone points.
3. The set of corner points, denoted as V_A, consists of all intersections of axes in A. It follows that A can be viewed as a graph with vertex set V_A. Denote by E_A the edge set.

Let X be a tubular closed neighborhood of F, consisting of a tubular neighborhood of A and $|B|$ small disks centered at B, such that $p(X)$ consists of a collar neighborhood of $\mathbf{S}^2/G$ and $|p(B)|$ small disks. Then both ∂X and $\partial p(X)$ consist of disjoint circles, so $e(\partial X) = e(\partial p(X)) = 0$. By the homotopy invariance, $e(X) = e(F) = |V_A| + |B| - |E_A|$, $e(p(X)) = |p(B)|$. Let $Y := \overline{\mathbf{S}^2 - X}$. By the lemma, $e(Y) = |G| e(p(Y))$. We have

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$$\begin{aligned} 2 = e(\mathbf{S}^2) = e(X) + e(Y) - e(\partial X) = |V_A| + |B| - |E_A| + e(Y)e(\mathbf{S}^2/G) \\ = e(p(X)) + e(p(Y)) - e(\partial p(X)) = |p(B)| + |G|^{-1}e(Y). \end{aligned}$$

To do further evaluations, denote by n_i $(i \in p(B))$ the orders of cone points, and by m_i $(i \in p(V_A))$ the orders of corner points. By the relation between orbit and stabilizer, each $i \in p(B)$ is an orbit of $\frac{|G|}{n_i}$ points. Thus $|B| = \sum_{i\in p(B)} \frac{|G|}{n_i}$. Similarly, $|V_A| = \sum_{i\in p(V_A)} \frac{|G|}{2m_i}$. Since a corner point $x \in V_A$ has order m if and only if its degree in the graph $d(x) = 2m$, we have $|E_A| = \frac{1}{2}\sum_{x\in V_A} d(x) = \sum_{x\in V_A} m_x = \sum_{i\in p(V_A)} \frac{|G|}{2m_i} \cdot m_i = \frac{|G||p(V_A)|}{2}$. Combining all equations obtained so far yields

$$\begin{aligned} e\left(\frac{\mathbf{S}^2}{G}\right) &= |p(B)| + \frac{2 - |V_A| - |B| + |E_A|}{|G|} \\ &= \sum_{i\in p(B)} 1 + \frac{2}{|G|} - \sum_{i\in p(V_A)} \frac{1}{2m_i} - \sum_{i\in p(B)} \frac{1}{n_i} + \frac{|p(V_A)|}{2} \qquad (1) \\ &= \frac{2}{|G|} + \sum_{i\in p(B)} \frac{n_i - 1}{n_i} + \sum_{i\in p(V_A)} \frac{m_i - 1}{2m_i}. \end{aligned}$$

In particular, $e(\mathbf{S}^2/G)$ is positive, which gives us only three possibilities: $S^2/G = S^2$, S^2_{-1} or P^2.

- $S^2/G = P^2$. It is boundaryless, so $A = \varnothing$. Then the equation writes

$$1 = \frac{2}{|G|} + \sum_{i\in p(B)} \frac{n_i - 1}{n_i}. \qquad (2)$$

Since $\frac{n_i-1}{n_i} \geq \frac{1}{2}$, $|p(B)| \leq 1$. The solutions are $(n_1, |G|) = (N, 2N)$ for $N \geq 2$. The orbifold notation is $N\times$, where $\times$ stands for a crosscap, and N refers to one cone point of order N. One remaining solution is $|G| = 2$, $p(B) = \varnothing$ (i.e. G generated by τ), which is denoted by $\times$ or $1\times$.

- $S^2/G = S^2_{-1}$. The equation then writes

$$1 = \frac{2}{|G|} + \sum_{i\in p(B)} \frac{n_i - 1}{n_i} + \sum_{i\in p(V_A)} \frac{m_i - 1}{2m_i}. \qquad (3)$$

Since $\frac{n_i-1}{n_i}, \frac{m_i-1}{m_i} \geq \frac{1}{2}$, we have $\frac{|p(B)|}{2} + \frac{|p(V_A)|}{4} < 1$.

(i) $|p(B)| = 1$, $|p(V_A)| = 1$. Then we have

$$\frac{1}{n_1} + \frac{1}{2m_1} = \frac{2}{|G|} + \frac{1}{2} > \frac{1}{2}.$$

The solutions can be enumerated: $(n_1, m_1, |G|) = (2, N, 4N)$ ($2*N$) and $(3,2,24)$ (3*2).

(ii) $|p(B)| = 1$, $|p(V_A)| = 0$ (which implies that A consists of a single great circle since the orbifold has boundary). The solution is $(n_1, |G|) = (N, 2N)$ for $N \in \mathbf{N}^*$ ($N*$).

(iii) $|p(B)| = 0$, $|p(V_A)| \leq 3$. We have

$$\frac{1}{2m_1} + \frac{1}{2m_2} + \frac{1}{2m_3} = \frac{2}{|G|} + \frac{1}{2} > \frac{1}{2}, m_1 \leq m_2 \leq m_3.$$

The solutions are: (m1, m2, m3, |G|) = (2, 2, N, 4N) ($*22N$), $(2,3,3,24)$, $(2,3,4,48)$, $(2,3,5,120)$(*332,*432,*532) and $(1, N, N, 2N)$ ($*NN$).

- $S^2/G = S^2$. So $A = \varnothing$ and the equation writes

$$2 = \frac{2}{|G|} + \sum_{i \in p(B)} \frac{n_i - 1}{n_i}. \tag{4}$$

We have $|p(B)| \leq 3$, then

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{2}{|G|} + 1 > 1,\ n_1 \leq n_2 \leq n_3.$$

The solutions are $(n_1, n_2, n_3, |G|) = (2,2,N,2N)$ (22N), $(2,3,3,12), (2,3,4,24), (2,3,5,60)$ (332,432,532) and $(1,N,N,N)$ (NN).

Remark. There is a subtle problem that, how can we recover the groups from the solutions? If we can show that the solutions NN, $22N$, 332, 432, 532 indeed correspond to C_N, D_N, T, O, I, then theorem 3.5 is proved and the result of $O(3)$ follows. To argue this, once the locations of the rotation centers is determined, the group is pinned down up to conjugation. The solution NN clearly corresponds to C_N. For $22N$, the rotation at 2 (resp. N) must send N (resp. 2) to N (resp. 2), so the graph presentation is unique up to conjugation. As for $k32$ ($k = 3,4,5$), the rotation of order 3 composed with that of order k must be the rotation of order 2, which determines the shape of a tetrahedron / octahedron / icosahedron.

Remark. It turns out that the proof of the Euler number formula

$$e\left(\frac{\mathbf{S}^2}{G}\right) = \frac{2}{|G|} + \sum_{i \in p(B)} \frac{n_i - 1}{n_i} + \sum_{i \in p(V_A)} \frac{m_i - 1}{2m_i} \tag{5}$$

can be simplified drastically once one has the background of branched covering. Here $p: \mathbf{S}^2 \to \mathbf{S}^2/G$ is a covering branched at the singularities. Choose a sufficiently fine triangulation such that all singularities are 0-simplexes and boundaries consist of 1-simplexes. Then $|G|e(\mathbf{S}^2/G) = 2$ approximately. Except that, at a cone point of order n on the orbifold, there are only $\frac{|G|}{n}$ points over it. So we need to subtract $|G| - \frac{|G|}{n}$ from the left. At the boundary circle p is $\frac{|G|}{2}$ to 1, but circle has Euler number 0 anyway. At a corner point of order m on the orbifold, there are only $\frac{|G|}{2m}$ points over it, so we need to subtract $\frac{|G|}{2} - \frac{|G|}{2m}$ from the left. Combining all the corrections, we get the formula. This is in principle very similar to the proof of Riemann-Hurwitz formula.

4. The Euclidean case

4.1. Wallpaper groups

A crystallographic group of $\mathbf{E}^2$ is called a wallpaper group. We have seen in section 2.3 that elements in $\mathrm{Iso}(\mathbf{E}^2)$ are rotations, reflections, glide reflections and translations. The first two have fixed points, and produce cone points and corner points on the orbifold if G is a wallpaper group. By Bieberbach's theorems, their order is at most 6, and there is a lattice of rank 2 in G. It turns out that this restricts G to finitely many possibilities. To give the classification, we have two methods, either we build on the work in section 2.3, or we try to transplant the proof of the case $\mathbf{S}^2$ over $\mathbf{E}^2$.[1] We begin with the first method.

The simplest case is that G is just the lattice, i.e. generated by 2 linearly independent vectors. So the fundamental domain is just a parallelogram, and the orbifold is a torus T^2. This group (up to conjugation) is denoted as $\circ$ in Conway's notation.

If, G contains some extra elements, then all of them have orthogonal part preserving the lattice Λ (recall lemma 2.12). So the question is: what are other symmetries Λ can have, besides the translations?

A first example is that, for a general parallelogram $ABCD$, there is a central symmetry (rotation by π) r at the center O. The compositions rt_{AB}, rt_{AD} and rt_{AC} give three more central symmetries r_{AD}, r_{AB}

and r_A. The group obtained this way, is denoted as 2222, as there are 4 cone points of order 2 on the orbifold.

To include more symmetries, the parallelogram $ABCD$ must have some speciality. For example, when it is a rhombus, then there are two reflection lines meeting at O. The vertices form a whole orbit, which is another corner point of order 2. Together, they form the group $2{*}22$. If we only include reflections lines of one direction and exclude all rotation symmetries, then we get its subgroup $*\times$, of index 2. The $\times$ stands for a glide reflection as composition of the reflection and translation.

Now we see the pattern. Start with Λ, list all its symmetries, then combine them in all different ways. Whenever we find some of them closed under the group law, we get a wallpaper group. It remains to do some careful inspection.

- Λ is a rectangle lattice. There are reflection lines of two directions. All of them form the wallpaper group $*2222$. Include only reflections of one direction and exclude all rotations, we get $**$., subgroup of $*2222$ of index 2. Include only reflections of one direction, as well as all rotations, we get $22*$. Both it and $*2222$ contains two subgroups of index 2, $**$ and 2222.
- Actually, $22*$ contains another subgroup of index 2, $\times\times$. The two $\times$ are the composition of $*$ with the two 2 respectively. Another group having $\times\times$ and 2222 as two subgroups of index 2 is $22\times$. For example, see below:

Figure 9. A symmetrical pattern of $22\times$.

- Λ is a square lattice. We get $*442$ first, consisting of reflection lines of four directions. The rotations in it form a subgroup of index 2, $*442$. Another group having 442 as subgroup of index 2 is obtained by including only reflections of two oblique directions, with notation $4{*}2$. For example, figure 1 and 2 are respectively $*442$ and $4{*}2$.
- Λ is a regular triangle lattice. Like the above, there are two pairs of index-2 subgroup, $*632$ and 632 (for example, figure 3), $*333$ and 333. Another subgroup of $*632$ of index 2 is $3{*}3$, consisting of rotations at the centers of all triangles, and reflections along their edges.

We have by far collected a total of 17 wallpaper groups. However, as we have seen, the above discussion is rather troublesome, and is very likely to leave out some cases hence is not very convincing. This is why we now turn to the second method, the orbifold classification. It can be viewed as a verification that our list of the 17 groups is indeed complete, as we will find out exactly 17 solutions of the equation.

We try to derive a formula of $e\left(\mathbf{E}^2/G\right)$, where G is a wallpaper group. Unlike the case $\mathbf{S}^2$, now we have infinitely many fundamental domains hence infinitely many singularities. To deal with this, we consider R_N, consisting of a total of N closed fundamental domains in a disk of radius r centered at the origin, and let $N \to \infty$ at the end. Note that N is proportional to r^2. Now $p_N: R_N \to \mathbf{E}^2/G$ is a N-sheeted covering branched at the cone points, corner points and boundary points. As in the remark at the end of the last chapter, $Ne\left(\mathbf{E}^2/G\right) = e(R_N) = 1$ approximately. If a singularity $x \in \mathbf{E}^2/G$ has stabilizer group of order k, then $\left|p_N^{-1}(x)\right| = \frac{N}{k} + O(r)$. This error term $O(r) = o(N)$ is due to the fact that some symmetries at ∂R_N are cut off. Recall that cone points and corner points on the orbifold are finitely many. We get the asymptotic formula

$$Ne\left(\frac{\mathbf{E}^2}{G}\right) = 1 + \sum_{\text{cones}} \left(N - \frac{N}{n_i}\right) + \sum_{\text{corners}} \left(\frac{N}{2} - \frac{N}{2m_i}\right) + o(N) \tag{6}$$

and letting $N \to \infty$ yields

$$e\left(\frac{\mathbf{E}^2}{G}\right) = \sum_{\text{cones}} \frac{n_i - 1}{n_i} + \sum_{\text{corners}} \frac{m_i - 1}{2m_i}. \tag{7}$$

In particular, $e(\mathbf{E}^2/G) \geq 0$, which gives us 7 possibilities as below:

- $\mathbb{R}^2/G = S^2$, solutions are 632, 442, 333, 2222.
- $\mathbb{R}^2/G = T^2$, so G is generated by two translations, that is ∘.
- $\mathbb{R}^2/G = P^2$, we have 22×.
- $\mathbb{R}^2/G = 2P^2$ (the Klein bottle), we have ××.
- $\mathbb{R}^2/G = S^2_{-1}$, we have *632, *442, *333, *2222, 2*22, 3*3, 4*2, 22*.
- $\mathbb{R}^2/G = S^2_{-2}$, we have **.
- $\mathbb{R}^2/G = P^2_{-1}$, we have *×.

Note that unlike the spherical case where a type can be an infinite family, here the 17 types are indeed 17 individual groups. So it is really a finite list of wallpaper groups.

4.2. Frieze patterns

See the below picture (from [2]), which is also symmetrical in some sense:

Figure 10. A symmetrical pattern on buildings.

Mathematically this is a crystallography on the infinite stripe $\mathbf{E} \times [0,1]$, called a frieze pattern. It is categorized as a Euclidean crystallography because the metric is flat. However, we will soon see that its classification is closely related to the classification of spherical crystallography.

First we study the isometries. An isometry f of $\mathbf{E} \times [0,1]$ will preserve its boundary, i.e. two copies of $\mathbf{E}$. Moreover, after $f|_{\partial(\mathbf{E}\times[0,1])}$ is determined, actually the whole f is determined. So the knowledge on isometries of $\mathbf{E}$ suffices for understanding the isometries of $\mathbf{E} \times [0,1]$. There is a special isometry τ, namely, the reflection along $\mathbf{E} \times \{\frac{1}{2}\}$. In general, there are 4 types of f:

- If f preserves each piece of boundary, then f is either a translation or a reflection. That is, an isometry of $\mathbf{E}$ times $id_{[0,1]}$.
- If f swaps two pieces of boundary, then $\tau \circ f$ belongs to the previous case. Then f can be a glide reflection (composition of τ and translation), or a central symmetry (composition of τ and reflection).

In particular, f^2 is either a translation of id. It is easy to show that every frieze group contains translations. Thus we can say that frieze patterns are periodic. And by quotienting out a period, we can wrap them along the equator of a sphere. Though this map is not isometric, it converts every frieze isometry to a spherical isometry:

- Translation corresponds to a rotation fixing two poles, and is denoted as ∞.

- Reflection corresponds to reflection along a longitude, denoted as ∞.
- Glide reflection corresponds to a glide reflection along the equator, denoted as $\times$, except τ is denoted as $*$.
- Central symmetry corresponds to a rotation of order 2, denoted as 2.

In this way, every frieze group is converted to a type of spherical crystallographic group. Hence conversely, it suffices to examine which among the 14 types can be restricted to a frieze group along the equator. They are exactly those with singularities only on the equator and at the two poles. We have 7 of them: NN, $N\times$, $N*$, $*NN$, $22N$, $*22N$ and $2*N$, presented in figure 10 by footprints, and called by their Conway's nicknames. For example, figure 9 is a $2*\infty$.

5. Hyperbolic geometry and quaternions

5.1. Quaternions and spherical rotations

We study a higher dimensional case, $\mathbf{S}^3$, in this section. Note that our method of orbifold classification as we treated the case $\mathbf{S}^2$ fails completely. The reason is twofold. First, the topology of 3-manifolds is wild, the Euler number does a very poor job on distinguishing the orbifolds, and there is no simple and satisfactory topological invariant. Second, even we manage to derive a formula like the Euler number formula it would be too complicated, since unlike the case $\mathbf{S}^2$ where we only have the cone points and corner points to worry about, now there are 14 types of singularities, corresponding to the 14 types of stabilizers as finite subgroups of $O(3)$. We must seek for another method. After reading [6] we know that the problem is related to quaternions, a generalization of complex numbers.

Consider the quaternions $q = a + bi + cj + dk$. The space of unit quaternions (i.e. those satisfying $a^2 + b^2 + c^2 + d^2 = 1$) can be identified with $\mathbf{S}^3$. Moreover, the subspace of those satisfying $a = 0$ is a $\mathbf{S}^2$. That is,

$$\mathbf{S}^2 = \{q \mid |q| = 1, q + \overline{q} = 0\}.$$

Proposition 5.1. For any $q \in \mathbf{S}^3$, $x \in \mathbf{S}^2$, we have $qxq^{-1} \in \mathbf{S}^2$. Moreover, the map $L_q: x \mapsto qxq^{-1}$ is an orientation-preserving isometry of $\mathbf{S}^2$.

Proof: Since $|qxq^{-1}| = |q||x||q^{-1}| = 1$ and $qxq^{-1} + \overline{qxq^{-1}} = q(x + \overline{x})q^{-1} = 0$, $qxq^{-1} \in \mathbf{S}^2$. For any $x, y \in \mathbf{S}^2$, we have

$$|L_q(x) - L_q(y)| = |q||x - y||q^{-1}| = |x - y|$$

so L_q is an isometry. It follows that $q \mapsto L_q$ defines a continuous group homomorphism p from $\mathbf{S}^3$ (group law given by the quaternion multiplication) to $O(3)$. Since $\mathbf{S}^3$ is path-connected, the image must be contained in $SO(3)$, the component of id. Thus $L_q \in SO(3)$.

Conversely, one can show that every rotation of $\mathbf{S}^2$ can be expressed in the form L_q (using the fact that a rotation is determined by its value at 2 points). If $L_q = id$, then q is in the center of quaternions, $\{\pm 1\}$. In conclusion, we have (see section 3, [6]:

Theorem 5.2. The homomorphism $p: \mathbf{S}^3 \to SO(3)$ defined above is a 2-1 covering map.

Another similar result is that (see section 4, [6]:

Theorem 5.3. For $q_1, q_2, x \in \mathbf{S}^3$, we have

$$L_{q_1,q_2}(x) := q_1 x q_2^{-1} \in \mathbf{S}^3.$$

Then $(q_1, q_2) \mapsto L_{q_1,q_2}$ defines a group homomorphism from $\mathbf{S}^3 \times \mathbf{S}^3$ to $SO(4)$. Moreover, it is a 2-1 covering map.

Therefore, the classification of finite subgroups of $O(4)$ is based on that of $O(3)$, and is roughly divided into the following steps: 1. pass from finite subgroups of $O(3)$ to those of $\mathbf{S}^3$; 2. pass from $\mathbf{S}^3$ to

$\mathbf{S}^3 \times \mathbf{S}^3$; 3. from $\mathbf{S}^3 \times \mathbf{S}^3$ to $SO(4)$; 4. from $SO(4)$ to $O(4)$. We only give an outline of each step as below. A complete list is in section 4, [2].

1. If $G \subset \mathbf{S}^3$ is finite, then $p(G) = C_n, D_n, T, O, I$. Since -1 is the only order 2 element in $\mathbf{S}^3$, if $2 \,|\, |G|$, then $-1 \in G$ hence $G = p^{-1}(p(G))$. It follows that $G = C_n, p^{-1}(D_n), p^{-1}(T), p^{-1}(O), p^{-1}(I)$.
2. If $G \subset \mathbf{S}^3 \times \mathbf{S}^3$ is finite, then the two projection images G_1 and G_2 belong to the above 5 types. By Goursat's lemma (see p75, 8), G is uniquely determined by normal subgroups N_1, N_2 of G_1, G_2 together with an isomorphism $G_1/N_1 \cong G_2/N_2$. That is, two finite subgroups of $\mathbf{S}^3$ with a common quotient.
3. Finite subgroups of $SO(4)$ are just the image of those of $\mathbf{S}^3 \times \mathbf{S}^3$.
4. As in section 3.2, a finite subgroup of $O(4)$ is determined by a finite subgroup $H \subset SO(4)$, together with an element $a \in O(4) - SO(4)$ such that $a^2 \in H$, $aH = Ha$.

$$D(p,q) := \langle a,b,c \mid a^q = b^p = c^2 = abc = id \rangle \ ^3$$

is infinite, if and only if $(p-2)(q-2) \geq 4$.

Note that $(p-2)(q-2) < 4$ if and only if $(p,q) = (2,n), (3,3), (3,4), (3,5)$. These correspond exactly to the groups D_n, T, O, I in the spherical case. The solutions of $(p-2)(q-2) = 4$ are $(3,6)$ and $(4,4)$, which correspond to the groups 632. and 442 in the Euclidean case. These groups consist of rotation symmetries of a platonic solid or a Euclidean tessellation. For the infinitely many cases $(p-2)(q-2) > 4$, the group $D(p,q)$ can be realized as the rotation group of a tessellation (denoted as $\{p,q\}$) by regular q-gons, such that at each vertex p regular gons meet. This tessellation is impossible in the usual Euclidean and spherical geometry, but can happen on the hyperbolic plane $\mathbf{H}^2$, thanks to the following result (theorem 3.5.5, [3]).

Theorem 5.5. On the standard Poincare disk model of $\mathbf{H}^2$, a triangle with interior angles α, β, γ has area $\pi - \alpha - \beta - \gamma$.

It follows that for a q-gon, the difference between $(q-2)\pi$ (the Euclidean interior angle sum) and its actual interior angle sum, is its hyperbolic area. Hence an interior angle of a regular q-gon ranges from $\left(1 - \frac{2}{q}\right)\pi$ to 0. To fit p copies of a regular q-gon into one corner, it suffices to take its interior angle to be $\frac{2\pi}{p}$, provided $(p-2)(q-2) > 4$. Repeating this pattern infinitely on the Poincare disk, we get the tessellation $\{p,q\}$, which is a symmetrical pattern of $\mathbf{H}^2$, with rotation group $D(p,q)$. Thus, all $D(p,q)$ when $(p-2)(q-2) > 4$ is a crystallographic subgroup of $\mathrm{Iso}(\mathbf{H}^2)$. In Conway's notation it is $qp2$, meaning that the orbifold (which is $\mathbf{S}^2$) has three cone points of order $2, p, q$ respectively. If we also take reflection symmetries into account, we get the crystallographic group $*qp2$, meaning that the orbifold (which is $\mathbf{S}^2_{-1}$) has three corner points of order $2, p, q$ respectively, which contains $qp2$ as a subgroup of index 2.

Example. The following is the tessellation $\{3,7\}$. The 7-gons are divided into 14 triangles, to better illustrate the symmetry $*732$.

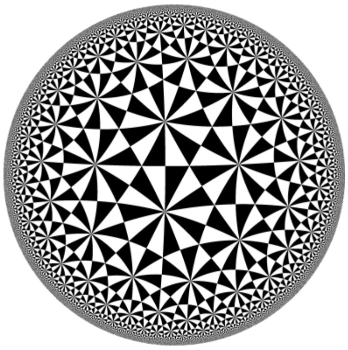

Figure 11. Taken from orbifold notation, wikipedia.

Alternatively, one can first draw the Cayley graph of $D(p,q)$ on the Poincare disk, then refine it into the picture above.

Recall that in the Euclidean case we proved the Euler number formula

$$e\left(\frac{\mathbf{E}^2}{G}\right) = \sum_{\text{cones}} \frac{n_i - 1}{n_i} + \sum_{\text{corners}} \frac{m_i - 1}{2m_i} \tag{8}$$

hence yielded finitely many solutions. It is also true that on $\mathbf{H}^2$, the isometries having fixed points are rotations and reflections. Why the formula above does not hold anymore? Looking closely into the proof, we realize that the fact that on a circle of radius r there are approximately $O(r)$ fundamental domains, is no longer true in hyperbolic geometry. This makes the formula fail for $\mathbf{H}^2$, as we can check for all the examples mentioned above that, the right hand side is greater than the left hand side.

An infinite family of tessellations we found particularly interesting is $\{4g, 4g\}$, $g \geq 2$. We explain it with the case $g = 2$:

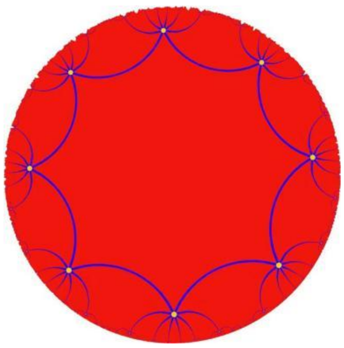

Figure 12. Taken from orbifold notation, wikipedia.

This reflects the fact that $\mathbf{H}^2$ is the universal cover of the genus 2 surface, $2T^2$, whose fundamental group is

$$\langle a_1, b_1, a_2, b_2 | [a_1, b_1][a_2, b_2] \rangle.$$

Just like a torus is obtained from curling a square, to obtain $2T^2$, we first take a regular octagon $ABCDEFGH$, then identify AB with CD, BC with DE, EF with GH, FG with HA. If we place the octagon on the Poincare disk, then the identification of AB with CD is realized by a dilation (which is a hyperbolic isometry) along the line of BC, and so on. At each vertex 8 edges meet, corresponding to the 8 loops $a_1^{\pm}, b_1^{\pm}, a_2^{\pm}, b_2^{\pm}$. It follows that by a group action on $\mathbf{H}^2$ of G generated by four dilations, the octagon becomes a fundamental domain. Since there is no singularity, $\mathbf{H}^2$ covers the orbifold $\mathbf{H}^2/G \cong 2T^2$, hence it is a universal cover. In the same way, for any $g \geq 2$ there is a crystallographic subgroup

$G \subset \mathrm{Iso}(\mathbf{H}^2)$ generated by $2g$ dilations, such that the orbifold is gT^2. This shows that in dimension 2 hyperbolic crystallography, the orbifolds can have arbitrarily high genus.

6. Conclusion

In this paper we have studied symmetrical patterns / crystallography in various geometries, which are usually dealt with under separated topics by other authors. Those are, planar wallpaper groups (the Euclidean case), finite subgroups of the orthogonal groups (the spherical case), and cocompact Fuchsian groups (the hyperbolic case). For the Euclidean case, we proved that there are exactly 17 crystallographic groups (up to isomorphism) in dimension 2, via classification according to orbifolds, or according to lattices alternatively. We also proved that there are 7 frieze patterns. For the spherical case, by deriving an equation from computing the Euler number of orbifolds, we obtained in total 14 families of finite subgroups of $O(3)$, and gave a lovely graphic presentation of their group actions on S^2 due to Conway. We also discussed the method of dealing with the problem in $O(4)$. For the hyperbolic case, unlike the previous two, we showed that even in dimension 2 the classification is wildly complicated and we explained the reason why it is so. We studied some examples emerging from hyperbolic tessellations and topology.

Acknowledgments

I came into contact with this topic two years ago. At that time I realized my interest in geography. By chance, I learned the concept of geodesic lines which regrading to geometry. After I found the interesting relationship between geometry and geography, I started read some materials of geometry. By reading a book named Symmetry. I aware of that the topic of crystallography is the fittest topic, I have the knowledge reserve of this topic, and the interesting can encourage me to continue this project. By concluding the theories and researching this essay finally complete.

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