

Continuity of all Algorithmic Functions on Left Numbers

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Abstract. A constructive Function is a computer program that transforms a Constructive Real Number (CRN) into another CRN. A left number is a program that generates an increasing sequence of rational numbers with an upper bound. We use algorithmic functions to generate CRNs and Left numbers. In this essay, we will prove the continuity of such functions on Left numbers in topological space with the base of right rays.

Keywords: CF (constructive function); algorithmic functions, CRN (constructive real number); Left number; Topological space.

1. Definitions

1.1. Definition 1 (Constructive Real Number)

A computer program to describe a number with two parts α , β . Here α produces a Cauchy sequence of rational numbers, and β generates a convergence regulator, which tells how fast the sequence converges. They were first introduced by Turing [1,2].

1.2. Definition 2 (Constructive Function)

an algorithm transforming CRN to CRN.

1.3. Definition 3 (Left number)

an increasing has algorithmically given sequence of rational numbers with an upper bound. Besides, there are two schools of Constructive Analysis: one follows the bishop-Bridges approach, and the other follows the Markov-Shanin approach [3-6].

1.4. Definition 4 (Topology)

Topology is the mathematical study of the properties that are preserved through deformations, twistings, and stretchings of objects. The study of topology is much more comprehensive than that of Euclidean geometry, and the notions of distances and points are much broader.

A more precise definition:

Let X be a non-empty set. A set T of subsets X is said to be a topology if:

(i) X and the empty set, \emptyset belongs to T

(ii) the union of any finite or infinite number of sets T belongs to T and

(iii) the intersection of any two sets T belongs to T
 The pair (X, T) is called a topological space.

1.5. Definition 5 (Topological base)

Given a set, a collection of subsets of the set is said to form a basis for a topological space or a basis for a topology if the following two conditions are satisfied:

1. The union of all members of the collection is the whole space
2. Any finite intersection of members of the collection is itself a union of members of the collection

The topology generated by this basis is the topology in which the open sets are precisely the unions of basis sets.

2. Introduction

Continuity demonstrates itself in different ways in different spaces. Before the formal proof of the result, there are three lemmas we need to prove. Lemma 1 and Lemma 2 together imply that all algorithmic functions on the space of Left Numbers are monotonic, and with Lemma 3 together, we can formulate the corollary:

Every algorithmic function $f: \mathcal{L} \rightarrow \mathcal{L}$ is continuous in the topology of the right rays.

Note: \mathcal{L} is the space of all left numbers.

Moreover, decidability and enumerability are two essential properties about sets. Subset A of the set of positive integers \mathbb{N}^+ is *decidable* if we can find an algorithm U , $\forall n \in \mathbb{N}^+$ to tell if $U(n)$ is inside A or not. Set A is called *enumerable* if we can generate an algorithm which prints out all the elements in A . There do exist examples of enumerable but undecidable sets; see Theorem 11 Shen and Vereshchagin [7-9].

3. Main Part

A left number is a strictly increasing algorithmically given sequence of rational numbers with a rational upper bound. Two left numbers can be added together but cannot be subtracted because the result would generally not be a Left number. Also, we can add and subtract a rational number from a left number. Moreover, we can define the relationship of $<$ and \leq on left numbers.

The relationship $L_1 < L_2$ is defined as

$$L_1 < L_2 \text{ if } \exists \frac{p_i}{q_i} \in L_2, \text{ s. t. } \frac{p_i}{q_i} > \text{ all } \frac{p_j}{q_j} \in L_1.$$

The relationship $L_1 \leq L_2$ is defined as

$$L_1 \leq L_2 \text{ if } \forall \frac{p_i}{q_i} \in L_1, \exists \frac{p_j}{q_j} \in L_2, \text{ s. t. } \frac{p_j}{q_j} > \frac{p_i}{q_i}.$$

In the definitions above the exists symbol means that the existing object can be found algorithmically.

Previously we have introduced the definition of Constructive Real Number (CRN). However, left numbers are not always CRNs. One such example comes from the famous Specker sequence. More precisely, consider the Specker sequence (*Kushner*, 1984), whose elements have standard regulators and choose an increasing subsequence of the rational entries. By definition of the Specker sequence, it would not admit a regulator. Recall that an algorithmically given sequence of natural numbers $\beta(n)$ is called the regulator of convergence of a CRN if

$$\text{if } \forall n, i, j, \quad i, j \geq \beta(n), \text{ we have } |\alpha(i) - \alpha(j)| < 2^{-n}.$$

3.1. Main Theorem (Monotonicity)

For every algorithmic function $f: \mathcal{L} \rightarrow \mathcal{L}$, if $a \geq b$, then $f(a) \geq f(b)$.

3.2. Corollary

Every algorithmic function $f: \mathcal{L} \rightarrow \mathcal{L}$ is continuous in the topology of the right rays.

(We will discuss the right ray topology later on.)

Remark: this is spiritually similar to the Markov-Tseitin theorem, saying that all constructive functions are continuous (Kushner, 1984).

The proof of the main theorem and the corollary follows from the combination of the lemmas below.

3.2.1. Lemma 1

For a given $f: \mathcal{L} \rightarrow \mathcal{L}$, $a \geq b \implies$ it is not true that $f(a) < f(b)$.

Here \mathcal{L} is the space of all Left numbers.

Proof:

Take a program $H: \mathbb{N}^+ \rightarrow \mathbb{N}^+$ that has an undecidable domain; therefore, we cannot algorithmically predict if the program will ever terminate on every given input.

We construct a left number P_n that till certain moment behaves as b and then jumps to behave as a . P_n satisfies that $P_n(k)=b(k)$ if H is still working on input n by step k .

Assume $H(n)$ terminated on step m , we look at $b(m)$. Because a is greater than or equal to b , we can algorithmically find a $s(m)$ such that $a_{s(m)} > b(m)$.

Put $P_n(m) = a_{s(m)} > b(m)$ and $P_n(m+k) = a_{s(m)+k}$

This sequence P_n satisfies the property of the left number because it is strictly increasing and ultimately has the same bound as left number a .

Now, we argue by contradiction and assume $f(a) < f(b)$, i.e., we can algorithmically find v in the sequence of rational numbers forming the left number $f(b)$, which is greater than all elements in $f(a)$.

We will create a program that detects if H will terminate on n or not. If this program exists, it contradicts the definition of H since H has an undecidable domain, but this constructed program provides a decision process. We will prove that there exists one such program.

Take n and launch two parallel processes with step alternation. On odd steps, we try to compute $H(n)$, and on even steps, we compute $f(P_n)$ and compare it to v .

If H is not applicable to n , which means that H will not terminate, then $f(P_n)$ is simply left number $f(b)$. Therefore, there will be a moment n when $f(P_n) > v$ and we can detect that moment.

If H is applicable to n , then the program we constructed will compute $H(n)$ and terminate. Therefore, the sequence P_n turns into an after a certain moment and thus, the elements in $f(P_n)$ never exceed v . We can check when H terminates and this, together with comparing the elements of $f(P_n)$ with v is the decision process.

However, according to the definition of H , a decision process cannot exist. Therefore, the assumption that $f(a) < f(b)$ is not true.

Q.E.D

3.2.2. Lemma 2

For left numbers a, b , if it is not true that $a > b$, then $a \leq b$.

It is remarked here that this Lemma 2 was independently proved by the participants in Hybrid 2021 Neoscholar Program.

Proof:

Because $a > b$ is not true, there doesn't exist an a_i bigger than all the elements in b . In order to prove that a is less than equal to b , we offer an algorithm which determines a b_i bigger than any given a_k .

We start from a_1 , check elements one by one in b until there is a $b_i > a_1$.

Then proceed to a_2 , we start from b_i to check which is bigger, if $b_i < a_2$, then continue to check b_{i+1} and so on until there is a $b_k > a_2$, if $b_i > a_2$ then b_k satisfies the condition.

We carry on with these steps, and we can always find a b_i bigger than a given a_k using this algorithm. Therefore, a is less than or equal to b .

Q.E.D

Main Theorem (Monotonicity):

For every algorithmic function $f: \mathcal{L} \rightarrow \mathcal{L}$, if $a \geq b$, then $f(a) \geq f(b)$.

Proof:

Substitute a, b in Lemma 2 with $f(a), f(b)$ and combine Lemma 1 and Lemma 2, we can prove the main theorem that function from left numbers to left numbers is monotonic.

Q.E.D

Before proving Lemma 3, we introduce how a left number compares with a rational number and the base of topology on the space of Left numbers.

Note: a right ray is an interval of Left numbers from a rational number to positive infinity.

Here we define how a left number compares with a rational number q .

Define a left number q , s.t

$$q_n = q - 2^{-n}, n \in \mathbb{N}^+$$

We say that a left number $L > q$ when $L > q_n$.

Define a set as

$$B = \{L \in \mathcal{L}, L \geq q, q \in \mathbb{Q}\}$$

This is a base element of the topology of right rays on the space of Left numbers because

1. The union of all B s constitutes the whole space of the Left number.

2. For right rays B_1, B_2 starting at q_1 and q_2 respectively and $x \in \mathcal{L}, x \in B_1 \cap B_2$, because $B_1 \cap B_2$ is a right ray, since there exists a B_3 starting at $\max\{q_1, q_2\}$, s.t

$$x \in B_3 = B_1 \cap B_2.$$

3.2.3. Lemma 3

For $a, b \in \mathcal{L}$, a is contained in every open neighborhood of b exactly when $a \geq b$.

Proof:

We first give an equivalent condition to $a \geq b$. Then, we claim that

$$a \geq b \Leftrightarrow \forall q \in \mathbb{Q}, \text{ if there is a member of } b > q, \text{ then there is a member of } a > q.$$

Here every rational number q is regarded as a left number by considering $q - \frac{1}{2^n}$.

We first prove this claim from right to left.

We can assign q to be b_{i+1} , then $b_i < q$.

Because

$$\exists a_j, \text{ s.t } a_j > q, a_j > b_{i+1} > b_i = q$$

So

$$\forall b_i, \text{ we can find an } a_j > b_i.$$

Then we prove from left to right.

Because $a \geq b$, we have

$$\forall b_i, \exists a_j > b_i.$$

Therefore, if there is a $b_i > q$, there exists an $a_j > q$.

Now we return to the original lemma.

First, we prove from right to left.

We try to show that a is contained in every open neighborhood of b .

Obviously, $b \in B$. We use the equivalent condition to $a \geq b$ here. Because there is a member of $b > q$, there is a member of $a > q$. Therefore $b > q$ and $a > q$, and $a \in B$.

Then we prove from left to right.

Without the loss of generality, we take an open neighborhood of b to be

$$B = \{L \in \mathcal{L}, L \geq q, q \in Q\}.$$

Then a is contained in this B . Here we use the equivalent condition to $a \geq b$ described in the beginning of Lemma's proof again. Because b belongs to B , we have $a > q$ and $b > q$. So, there is always a member of the left number a bigger than q if there is a member of the left number b bigger than q , which is equivalent to $a \geq b$.

Q.E.D

3.3. Corollary

Every algorithmic function $f: \mathcal{L} \rightarrow \mathcal{L}$ is continuous in the topology of right rays.

Proof:

The continuity of function in topological space is equivalent to the claim that the preimage of an open set is open.

Assume

$$L^\circ \in f^{-1}(L') \in B.$$

Then by monotonicity of f ,

$$f^{-1}(\{L \geq L'\}) \subset \{L \geq L^\circ\},$$

which is open by the claim in the proof of Lemma 3.

Moreover, we prove that there are no $L \in \{L \geq L^\circ\}$ which are not in $f^{-1}(\{L \geq L'\})$ and thus

$$f^{-1}(\{L \geq L'\}) = \{L \geq L^\circ\}.$$

Indeed, assume such L exists and take one such L^* .

By monotonicity, we have

$$f(L^*) \geq L'.$$

So,

$$L^* \in f^{-1}(\{L \geq L'\}).$$

We have a contradiction.

Q.E.D

Acknowledgement

This research was done during the Hybrid Program organized by Neoscholar company. Viktor Chernov and Vladimir Chernov created research projects.

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