Proceedings of the 3rd International Conference on Computing Innovation and Applied Physics
DOI: 10.54254/2753-8818/30/20241017

Beyond the Finite: An Exploration of Infinite-Dimensional Vector Spaces

Jenny Zhu

The Pennington School, Pennington, New Jersey, 08534, USA

zxnnkj77@gmail.com

Abstract. In this paper, we delve deeply into the intricacies of linear algebra, with a focus on the progression from finite to infinite-dimensional vector spaces. Starting with the foundational concepts, we define vectors, vector spaces, linear combinations, and basis. The importance of infinite-dimensional vector spaces is emphasized, particularly their role in better understanding and modeling complex mathematical phenomena. Through well-illustrated examples, we guide the reader on how to validate if a given set can be classified as a vector space. Additionally, the methodology to identify bases for these vast spaces is discussed in detail. Reduction methods also play an important role in determining bases for infinite-dimensional spaces. In our conclusion, we reflect on the evolution from basic vector concepts to the more nuanced understanding of infinite dimensions. This progression not only deepens our understanding of vectors but also sets the stage for advanced investigations into linear relationships and transformations. By bridging the gap between elementary vector knowledge and advanced infinite-dimensional spaces, this paper makes a notable contribution to the ever-evolving field of linear algebra, serving as a valuable resource for both students and practitioners.

Keywords: Linear Algebra, Vector Space, Infinite Dimensions

1. Introduction

Vector spaces are a fundamental concept in mathematics that provides a powerful framework for studying linear relationships and transformations. This field encompasses a wide range of topics, from basic vector arithmetic to advanced linear algebra and functional analysis. Vector spaces not only find extensive use in pure mathematics but also play a crucial role in various scientific and engineering disciplines, making them valuable tools for mathematicians, physicists, engineers, and researchers from diverse fields[1].

Vector spaces are non-empty sets of elements. In the conventional understanding of vector spaces, these elements are finite-dimensional[2]. However, as mathematical inquiries and real-world challenges became more sophisticated, the limitations of finite-dimensional vector spaces became evident. There emerged a need to explore more intricate structures that could accurately model continuous and infinitely varying phenomena, which gave rise to the concept of infinite-dimensional vector spaces. Infinite-dimensional vector spaces deal with sets, including sets of functions, enabling mathematicians and scientists to study complex systems like Hilbert spaces, Banach spaces, partial differential equations, and quantum mechanics more rigorously. The extension to infinite dimensions

DOI: 10.54254/2753-8818/30/20241017

is also essential in functional analysis, a field that facilitates a deeper understanding of linear functions and their properties[3].

2. Definitions in Finite-Dimensional Vector Space

To better understand the concept of infinite-dimensional vector spaces, let us begin by defining what a vector space is.

2.1. Vector Space

In this section, we define a vector space as a non-empty set V that satisfies the following ten conditions for vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and scalars $a, b \in \mathbb{R}$:

1. The set V is closed under vector addition: $\vec{u} + \vec{v} \in V$.
2. Vector addition is commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. Vector addition is associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
4. There exists a zero vector $\vec{0} \in V$ such that $\vec{u} + \vec{0} = \vec{u}$ for all $\vec{u} \in V$.
5. Each $\vec{u} \in V$ has an additive inverse $\vec{v} \in V$ such that $\vec{u} + \vec{v} = \vec{0}$.
6. The set V is closed under scalar multiplication: $a \cdot \vec{u} \in V$.
7. Scalar multiplication distributes over scalar addition: $(a + b) \cdot \vec{u} = a \cdot \vec{u} + b \cdot \vec{u}$.
8. Scalar multiplication distributes over vector addition: $a \cdot (\vec{u} + \vec{v}) = a \cdot \vec{u} + a \cdot \vec{v}$.
9. Ordinary multiplication of scalars with scalar multiplication: $(ab) \cdot \vec{u} = a \cdot (b \cdot \vec{u})$.
10. Multiplication by the scalar 1 is the identity operation: $1 \cdot \vec{u} = \vec{u}$.

While finite-dimensional vector spaces are well-understood and are taught as the foundation of linear algebra, the principles that govern them serve as the bedrock for infinite-dimensional spaces. Just as the definition of a finite-dimensional vector space rests on vectors, linear combinations, and bases, infinite-dimensional spaces extrapolate these notions to encompass a broader array of mathematical objects, especially functions[4].

2.2. Vector

With vectors as the elements of the space, let us quickly define vectors as well. Here, instead of a geometric object that has magnitude (or length) and direction in $\mathbb{R}^2$ or $\mathbb{R}^3$, we generalize the concept of **vector** to simply an element of a vector space, following the 10 properties of the vector space[4].

Now, after knowing what vectors and vector spaces are, it is time to shift focus to some other important concepts in Linear Algebra.

2.3. Linear Combination

A **linear combination** is a mathematical operation that involves multiplying a set of values by coefficients and then adding the results together. In the context of vectors or functions, a linear combination is formed by scaling each vector or function by a certain factor (coefficient) and then adding them together.

Mathematically, given a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and corresponding coefficients $a_1, a_2, \ldots, a_n$, the linear combination is expressed as:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n \tag{2.1}$$

For instance, in a two-dimensional space, you may have two vectors $\vec{v}_1 = [2,3]$and $\vec{v}_2 = [1, -1]$. Their linear combination with coefficients $a_1 = 2$ and $a_2 = -1$ would be:

$$2\begin{bmatrix}2\\3\end{bmatrix} + (-1)\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}4-1\\6+1\end{bmatrix} = \begin{bmatrix}3\\7\end{bmatrix} \tag{2.2}$$

DOI: 10.54254/2753-8818/30/20241017

2.4. Linear Independence

In linear algebra, a set of vectors is said to be **linearly independent** if none of the vectors in the set can be written as a linear combination of the others[2]. In other words, a set of vectors $\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n$ is linearly independent if the only solution to the "Equation 2.3" is the trivial solution where all coefficients $a_1, a_2, \ldots, a_n$ are zero:

$$a_1\overrightarrow{v}_1 + a_2\overrightarrow{v}_2 + \ldots + a_n\overrightarrow{v}_n = 0 \tag{2.3}$$

For instance, in $\mathbb{R}^3$ (the plane of real numbers), the vectors [1,0,0], [0,1,0] and [0,0,1] are linearly independent because neither can be formed by scaling or adding the other. On the other hand, the vectors $[1,2,-3]$ and $[2,4,-6]$ are linearly dependent because the second is just twice the first. Expanding the equation gives us a linear combination:

$$\begin{aligned} a_1 + 2a_2 = 0 \\ 2a_1 + 4a_2 = 0 \\ -3a_1 - 6a_2 = 0 \end{aligned} \tag{2.4}$$

Here, it is noticeable that "Equation 2.5" and "Equation 2.6" are both related with "Equation 2.4" by scalar 2 and -3, which results in one unique equation:

$$a_1 + 2a_2 = 0 \tag{2.5}$$

Solving this equation for a_1 gives:

$$a_1 = -2a_2 \tag{2.6}$$

Since we can find a non-trivial solution for a_1 and a_2 such that $a_1\overrightarrow{v}_1 + a_2\overrightarrow{v}_2 = 0$, the vectors $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ are indeed linearly dependent. Specifically, the vector $\overrightarrow{v}_2$ is simply twice the vector $\overrightarrow{v}_1$, which means it can be obtained by scaling the first vector[4].

2.5. Spanning

If a set of vectors $\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n$ **span** V, it means that any vector within V can be expressed as a linear combination of the vectors in the set:

$$[V] = \{a_1\overrightarrow{v}_1 + a_2\overrightarrow{v}_2 + \ldots + a_m\overrightarrow{v}_m \,|\, a_1, a_2, \ldots, a_m \in \mathbb{R} \text{ and } \overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_m \in V\} \tag{2.7}$$

Returning to our example in $\mathbb{R}^3$, the vectors [1,0,0], [0,1,0]and [0,0,1] span the entire plane, as you can reach any point in the plane by scaling and adding these two vectors together[4].

2.6. Basis

In the context of vector spaces, a **basis** is a fundamental concept that plays a crucial role in understanding the structure and dimensionality of the space. In linear algebra, a basis of a vector space V is a set of vectors that spans V and is linearly independent[2].

We call the sets of vectors like [1,0,0], [0,1,0] and [0,0,1] which span $\mathbb{R}^3$ and are linearly independent natural bases, where each vector in the basis has all components equal to zero except for one component which is equal to one[4].

2.7. Dimension

The concept of dimension is intrinsically tied to basis. The **dimension** of a vector space V is defined as the number of vectors in any basis of V.

For instance, in $\mathbb{R}^3$(our conventional three-dimensional space), a basis consists of three vectors, such as the natural basis $\{[1,0,0], [0,1,0], [0,0,1]\}$. So, the dimension of $\mathbb{R}^3$ is 3. It is noteworthy that all bases for a specific vector space will have the same number of vectors[4].

DOI: 10.54254/2753-8818/30/20241017

3. Expanding From Finite to Infinite-Dimensional Vector Spaces

The definition of vectors and vector spaces that we delved into in the previous section first materialized in the 19th century. Mathematicians such as Grassmann and Hamilton were instrumental in this pioneering phase[5]. Their contributions didn't merely represent abstract mathematical endeavors; they fundamentally reshaped the foundations upon which modern physics and engineering were built.

Grassmann developed the foundation for what is today recognized as linear algebra and vector spaces. He innovated the concept of the exterior product, expanding the realm of multiplication to vectors and paving the way for the exploration of higher-dimensional spaces. Hamilton, on the other hand, introduced quaternions — a type of non-commutative number system that extended complex numbers and has since found applications in a plethora of fields[5].

The vectors we discuss here, however, typically refer to those residing in Euclidean spaces, a cornerstone of geometric concepts we encounter daily. The allure of Euclidean spaces is their tangible, intuitive nature. They provide a framework where abstract mathematical constructs seamlessly translate to our observable reality. Euclidean spaces, denoted as $\mathbb{R}^n$, are where vectors have a clear, finite number, n, of components. For instance, the $\mathbb{R}^2$ plane or the three-dimensional space denoted by $\mathbb{R}^3$ epitomizes this[6]. Within these spaces, vectors are often visualized as arrows originating from one point to another, with length and direction, offering tangible representations of mathematical construction.

Consider an example vector $\overrightarrow{v}$ inhabiting $\mathbb{R}^2$ with components $v_1 = 2$ and $v_2 = 3$. This vector can also be represented as $\overrightarrow{v} = [2,3]$. To visually grasp this vector, Mathematica aids us by plotting an arrow in "Figure 1" extending from the origin to the point defined by these components[7].

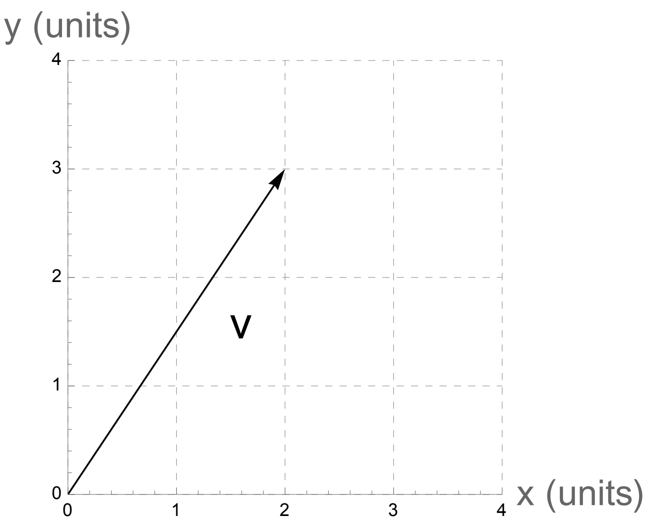

Figure 1. A visual representation of the vector $\overrightarrow{v} = [2,3]$ in $\mathbb{R}^2$. The arrow extends from the origin (0,0) to the point (2,3).

Similarly, venturing into $\mathbb{R}^3$, we encounter the vector $\overrightarrow{w}$ defined by its components $w_1 = 1$, $w_2 = -2$ and $w_3 = 4$. Expressed by $\overrightarrow{w} = [1, -2,4]$, we can portray $\overrightarrow{w}$ as an arrow initiating at the origin and culminating at the point dictated by its components in "Figure 2"[7].

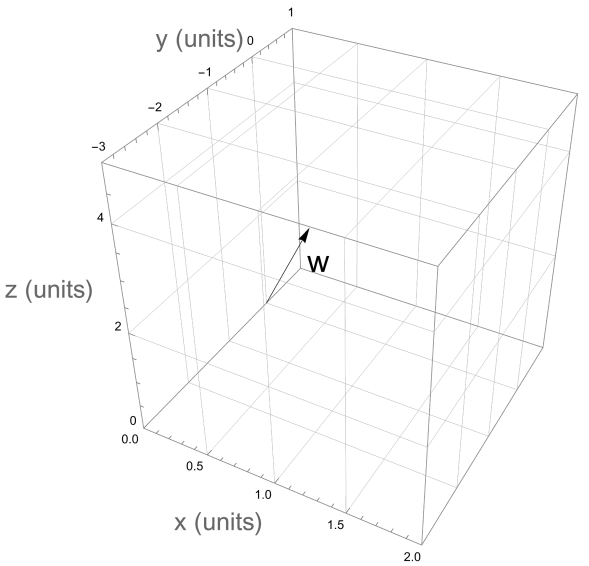

Figure 2. Three-dimensional visualization of the vector $\overrightarrow{w} = [1, -2,4]$ in $\mathbb{R}^3$ with the arrow originates from the origin and reaches the point $(1, -2,4)$.

While vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ provide a tangible representation, the real world, with its myriad of complexities, sometimes demands more. Phenomena governed by differential equations, for instance, often require a broader, more flexible framework than finite-dimensional spaces can offer. This realization precipitated the leap to infinite-dimensional spaces, primarily driven by the scrutiny of differential equations and the expanse of function spaces[8]. As the 20th century dawned, functional analysis burgeoned, positioning infinite-dimensional spaces at its nucleus. Herein, functions ascended from being merely confined to finite sets of variables, metamorphosing into vectors within these boundless spaces, and differential and integral operators as the transformations governing them[8].

David Hilbert, a luminary of this era, was foundational in fostering the study of infinite-dimensional vector spaces. His innovations weren't just expansions; they epitomized a paradigm shift. By recognizing functions as vectors, Hilbert equipped mathematicians and scientists with an unprecedented tool, deepening our insight into intricate mathematical frameworks and spawning novel perspectives on both physical and mathematical phenomena[6][10].

4. Examples in Infinite Dimensions

To have a better understanding of not only vectors but also the concept of infinite dimensions, let's start by proving the following are vector spaces and show that they are infinite-dimensional.

4.1. Prove that $\{(a_1, a_2, \ldots) \,|\, a_i \in \mathbb{R} \textit{ for all } i \in \mathbb{N} \textit{ and } \sum_{i=1}^{\infty} |a_i| < \infty\}$.

Proof. Let V be the set of vectors defined as:

$$V = \{(a_1, a_2, \ldots) \,|\, a_i \in \mathbb{R} \text{ for all } i \in \mathbb{N}\} \tag{4.1}$$

Let $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in V$ and scalar $b \in \mathbb{R}$ (to avoid confusion since a is mentioned in the problem), suppose $\overrightarrow{u} = (u_1, u_2, \ldots)$, $\overrightarrow{v} = (v_1, v_2, \ldots)$ and $\overrightarrow{w} = (w_1, w_2, \ldots)$. Here, we define vector addition and scalar multiplication of V to be:

1. $\overrightarrow{u}$ and $\overrightarrow{v}$ to be $\overrightarrow{u} + \overrightarrow{v} = (u_1 + v_1, u_2 + v_2, \ldots)$
2. $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ to be $\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w} = (u_1 + v_1 + w_1, u_2 + v_2 + w_2, \ldots)$
3. $c\overrightarrow{u} = (cu_1, cu_2, \ldots)$

To show that V is a vector space, we must prove that it satisfies the following properties of the vector space:

· **Closure under addition:** Following the definition of vector addition, $\overrightarrow{u} + \overrightarrow{v} = (u_1 + v_1, u_2 + v_2, \ldots)$. Since $u_i, v_i \in \mathbb{R}$ for all $i \in \mathbb{N}$, it follows that $u_i + v_i \in \mathbb{R}$. Also, $\sum_{i=1}^{\infty} |u_i + v_i| \le \sum_{i=1}^{\infty} |u_i| + \sum_{i=1}^{\infty} |v_i| < \infty$. Therefore, $\overrightarrow{u} + \overrightarrow{v} \in V$ and V is closed under addition.

· **Vector addition is commutative:** Due to the same definition, vector addition of $\overrightarrow{u}$ and $\overrightarrow{v}$ follows $\overrightarrow{u} + \overrightarrow{v} = (u_1 + v_1, u_2 + v_2, \ldots) = (v_1 + u_1, v_2 + u_2, \ldots) = \overrightarrow{v} + \overrightarrow{u}$. Since $u_i, v_i \in \mathbb{R}$ for all $i \in \mathbb{N}$, it follows that $u_i + v_i = v_i + u_i = k$ and $k \in \mathbb{R}$. Thus, $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$, the vector addition is commutative.

· **Vector addition is associative:** Similarly, we define vector addition of $\overrightarrow{u}$, $\overrightarrow{v}$and $\overrightarrow{w}$ follows: $\overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots)$. Then $(\overrightarrow{v} + \overrightarrow{u}) + \overrightarrow{w} = ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots)$. Since $u_i, v_i, w_i \in \mathbb{R}$ for all $i \in \mathbb{N}$, it follows that $u_i + (v_i + w_i) = (u_i + v_i) + w_i = k$ and $k \in \mathbb{R}$. Thus, $\overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{v} + \overrightarrow{u}) + \overrightarrow{w}$, the vector addition is associative.

· **Closure under scalar multiplication:** Using our definition of scalar multiplication, $c\overrightarrow{u} = (cu_1, cu_2, \ldots)$. Since $u_i \in \mathbb{R}$ for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$, it follows that $cu_i \in \mathbb{R}$. Moreover, $\sum_{i=1}^{\infty} |cu_i| = |c| \sum_{i=1}^{\infty} |u_i| < \infty$ Thus, $c\overrightarrow{u} \in V$, and V is closed under scalar multiplication.

· **Existence of additive inverses:** For any vector $\overrightarrow{u}$ in V, the additive inverse $-\overrightarrow{u} = (-u_1, -u_2, \ldots)$ is in V. We have $\overrightarrow{u} + (-\overrightarrow{u}) = (u_1 + (-u_1), u_2 + (-u_2), \ldots) = (0,0,\ldots)$, therefore exist additive inverses.

· **Satisfying other vector space properties:** The other vector space properties, such as ordinary multiplication of scalars with scalar multiplication, can be proven with the properties of $\mathbb{R}$.

Therefore, we have shown that V satisfies all the vector space properties, and hence, it is a vector space.

4.2. Prove that $\{f(x) \mid \int_{\mathbb{R}} |f(x)|\,dx < \infty\}$ *is a vector space and is infinite-dimensional*

Proof. To prove that the set $V = \{f(x) \mid \int_{\mathbb{R}} |f(x)|\,dx < \infty\}$ forms a vector space, we need to verify that it satisfies the vector space properties:

· **Closure under addition:** Let $f(x)$ and $g(x)$ be arbitrary functions in V. We have and be arbitrary functions in V so we have:

$$|f(x) + g(x)|\,dx \le \int_{\mathbb{R}} f(x)\,dx + \int_{\mathbb{R}} g(x)\,dx < \infty \tag{4.2}$$

Thus, $f(x) + g(x)$ also belongs to V.

· **Closure under scalar multiplication:** Let $f(x)$ be a function in V and let c be a scalar. We have be a function in and let be a scalar. We have

$$\int_{\mathbb{R}} |cf(x)|\ dx = |c| \int_{\mathbb{R}} f(x) < \infty \tag{4.3}$$

Therefore, $cf(x)$ is also in V.

· **Associativity, commutativity, and other properties:**The associativity of addition, commutativity of addition, and commutativity of scalar multiplication can be easily verified based on the properties of addition and scalar multiplication for functions. The existence of additive inverses and the compatibility of scalar multiplication with field multiplication can also be straightforwardly verified using the properties of addition and scalar multiplication for functions.

Since the set V satisfies all the vector space properties, we can conclude that V forms a vector space. This follows the properties of addition and scalar multiplication in $\mathbb{R}$.

The example above is known as the space of the L^1 function. For a function $f(x)$ defined on a certain interval, the norm of $f(x)$, denoted as $\|f\|_1$, is defined as the integral of the absolute value of the function over that interval. In mathematical notation:

$$\|f\|_1 = \int |f(x)|\, dx < \infty \tag{4.4}$$

On the other hand, we have the L^2 norm, also called the Euclidean norm, which quantifies the magnitude of a function by measuring the "spread" of its graph using the squared values of the function over the interval. Denoted as $\|f\|_2$, it corresponds to the square root of the integral of squared function values:

$$\|f\|_2 = \int |f(x)|^2\, dx < \infty \tag{4.5}$$

In comparison to L_1 functions, L_2 functions give greater weight to larger values and are sensitive to outliers, thus providing a measure of "total squared magnitude". These two norms offer distinct ways to assess the "size" of functions, each finding its applications in different areas of mathematics, such as optimization, signal processing, and image analysis. functions, functions give greater weight to larger values and are sensitive to outliers, thus providing a measure of "total squared magnitude". These two norms offer distinct ways to assess the "size" of functions, each finding its applications in different areas of mathematics, such as optimization, signal processing, and image analysis[9].

4.3.Find two bases for the vector space $V = \{a_1, a_2, \dots \mid a_i \in \mathbb{R}$ for all $i \in \mathbb{N}\}$ including a natural basis and the one that includes $[2,2,2\dots]$ *and* $[5,0,1,0,0\dots]$.

· **Natural Basis:** A natural choice for a basis is the set of sequences where only one entry is 1, and all others are 0. Specifically, for each natural number i let e_i be the sequence such that the i-th entry is 1 and all others are 0. This gives:

$$\begin{aligned} \overrightarrow{e}_1 &= [1,0,0,\dots] \\ \overrightarrow{e}_2 &= [0,1,0,\dots] \\ \overrightarrow{e}_3 &= [0,0,1,\dots] \\ &\vdots \end{aligned} \tag{4.6}$$

Thus, each element of V can be uniquely represented as a linear combination of the e_i making this set a basis for V.

· **Second Basis:** Given the vectors $\overrightarrow{u} = [2,2,2,\dots]$and $\overrightarrow{v} = [5,0,1,0,0,\dots]$we can consider them along with the natural basis vectors. However, both $\overrightarrow{u}$ and $\overrightarrow{v}$ can be expressed in terms of the natural basis e_i as:

$$\begin{aligned} \overrightarrow{u} &= 2(\overrightarrow{e}_1 + \overrightarrow{e}_2 + \overrightarrow{e}_3 + \dots) \\ \overrightarrow{v} &= 5\overrightarrow{e}_1 + \overrightarrow{e}_3 \end{aligned} \tag{4.7}$$

So, both $\overrightarrow{u}$ and $\overrightarrow{v}$ are linearly dependent on the natural basis e_i. If we want to include $u$and v in a basis, we would have to remove some of the e_i vectors, specifically e_1 and e_3 to avoid redundancy by applying Gauss's Method of reduction:

$$\begin{aligned} \overrightarrow{e}_1 &= \frac{1}{4}\left(\overrightarrow{v} - \frac{1}{2}(\overrightarrow{u} - 2(\overrightarrow{e}_2 + \overrightarrow{e}_4 + \overrightarrow{e}_5 + \dots))\right) \\ &= \frac{1}{4}\left(5\overrightarrow{e}_1 + \overrightarrow{e}_3 - (\overrightarrow{e}_1 + \overrightarrow{e}_3)\right) \\ &= \frac{4\overrightarrow{e}_1}{4} \\ &= \overrightarrow{e}_1 \end{aligned} \tag{4.8}$$

We can now use "Equation 4.8" for e_1 to reduce e_3:

DOI: 10.54254/2753-8818/30/20241017

$$\vec{e}_3 = \vec{v} - 5\vec{e}_1 \tag{4.9}$$

Thus, one possible basis that includes $\vec{u}$ and $\vec{v}$ can be:

$$\{\vec{u}, \vec{v}, \vec{e}_2, \vec{e}_4, \vec{e}_5, \ldots\} \tag{4.10}$$

By applying Gauss's Method of reduction, we have excluded e_1 and e_3. However, there are multiple bases that include u and v. The specific basis you choose may vary depending on the particular requirements or context of a problem.

We can also verify our result by proving $\{\vec{u}, \vec{v}, \vec{e}_2, \vec{e}_4, \vec{e}_5, \ldots\}$ as a basis for the vector space V.

Proof. To prove that the set $\{\vec{u}, \vec{v}, \vec{e}_2, \vec{e}_4, \vec{e}_5, \ldots\}$ is a basis for the vector space V, we need to show two properties:

· **Spanning Property:** We want to demonstrate that any vector $[a_1, a_2, \ldots] \in V$ can be expressed as a linear combination of the vectors in the given set. Given vectors $\vec{u} = [2,2,2,\ldots]$ and $\vec{v} = [5,0,1,0,0,\ldots]$, we consider an arbitrary vector $\vec{w} = [w_1, w_2, \ldots] \in V$which can be represented in terms of $\vec{u}$, $\vec{v}$, and the remaining basis vectors with scalar $c_u, c_w, c_2, c_4, c_5, \ldots$:

$$c_u\vec{u} + c_v\vec{v} + c_2\vec{e}_2 + c_3\vec{e}_3 + c_4\vec{e}_4 + \ldots = \vec{w} \tag{4.11}$$

A linear combination can also be derived with "Equation 4.11":

$$\begin{aligned} 2c_u + 5c_v &= w_1 \\ 2c_u + 2c_2 &= w_2 \\ 2c_u + c_v &= w_3 \\ c_4 &= w_4 \\ c_5 &= w_5 \\ &\vdots \end{aligned} \tag{4.12}$$

Solving for the scaler, we note that:

$$\begin{aligned} c_v &= \frac{w_3 - w_1}{4} \\ c_u &= \frac{3w_3}{8} + \frac{w_1}{8} \\ c_2 &= w_2 - \frac{3w_3}{4} - \frac{w_1}{4} \\ c_4 &= w_4 \\ c_5 &= w_5 \\ &\vdots \end{aligned} \tag{4.13}$$

This shows that we can express any vector $$w \in V$$ as a linear combination of the vectors in the given set, confirming the spanning property.

· **Linear Independence Property:** Let "Equation 4.11" equals 0 instead of $\vec{w}$:

$$c_u\vec{u} + c_v\vec{v} + c_2\vec{e}_2 + c_3\vec{e}_3 + c_4\vec{e}_4 + \ldots = 0 \tag{4.14}$$

We note that the only solution to this new "Equation 4.14" is when scalar $c_i = 0$, and thus each vector in the set is linearly independent.

As we have established both the spanning and linear independence properties, we conclude that the set $\{\vec{u}, \vec{v}, \vec{e}_2, \vec{e}_4, \vec{e}_5, \ldots\}$ forms a basis for the vector space V.

DOI: 10.54254/2753-8818/30/20241017

5. Conclusion

Our exploration has traced a transformative path from elementary vectors to the abstract realm of vector spaces, ultimately expanding our understanding to embrace infinite dimensions. Commencing with the elemental notion of individual vectors representing magnitude and direction, we ventured into the structured domain of vector spaces. These spaces provided a systematic framework, facilitating the investigation of linear relationships and transformations.

However, as the intricacies of continuous and intricate phenomena unfolded, the constraints of finite-dimensional vector spaces came to light. This compelled us to embark on a paradigm shift towards infinite-dimensional vector spaces. In this expanded context, the conventional rules of vector addition and scalar multiplication persevered, albeit with an evolved scope[10].

The progression into infinite dimensions was principally motivated by the study of differential equations and the exigencies of function spaces. The advent of functional analysis was instrumental, allowing functions themselves to be treated as vectors in these boundless spaces. This profound insight enabled a more accurate representation and examination of continuous phenomena. While finite-dimensional vector spaces excelled in discrete scenarios, infinite-dimensional vector spaces proved paramount for comprehending the nuanced intricacies of the continuous world.

Acknowledgments

I would like to express my sincere gratitude to my mentor, Elena Kim, for her invaluable guidance and unwavering support throughout the process of completing this essay. Her expertise and insights have been instrumental in shaping the direction of my work.

I am also deeply thankful to Jeffrey Shen, Lucy Zhang, and Ryan Vogt for their generous contributions. Their thoughtful feedback and constructive suggestions significantly enhanced the quality of this paper.

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