Exploring projective equivalences between closures of orbits

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Abstract. Motivated by results in the literature that use representations and group actions to produce nice geometric results about algebraic varieties, this article studies projective equivalence relations between closures of orbits for several complex algebraic group actions G on $\mathbb{P}(V)$, where V is a complex representation of G. In particular, we study the cases when (G, V) is one of the following: $(GL_n(\mathbb{C}), \mathbb{C}^n)$, $(O_n(\mathbb{C}), \mathbb{C}^n)$, $\big(GL_n(\mathbb{C}), M_n(\mathbb{C})\big)$, and $\big(O_n(\mathbb{C}), M_n(\mathbb{C})\big)$. On the way, we also obtain some interesting geometric results from studying these orbits.

Keywords: projective equivalence, orbits, representations of algebraic groups.

1. Introduction

Motivated by studies of orbits of algebraic group actions [1] and their compactifications [2] with different properties, we propose to explore the following problem.

Problem 1. Let G be a linear algebraic group defined over the field $\mathbb{C}$ of complex numbers. Let V be a complex algebraic representation of G. Then G acts canonically on $\mathbb{P}(V)$. Let $x \in \mathbb{P}(V)$ and consider the orbit of x under the action G on $\mathbb{P}(V)$, which we denote by G_x. We examine the following questions.

(i) When is G_x projectively equivalent to $G_{x'}$?

(ii) Let $\Omega_x := \overline{G_x}$ be the closure of G_x in $\mathbb{P}(V)$. Under what conditions on $x, x' \in \mathbb{P}(V)$ are Ω_x and $\Omega_{x'}$ projectively equivalent?

In this article, we study this problem for the following four cases.

- $G = GL_n(\mathbb{C})$ is the general linear group over $\mathbb{C}$, $V = \mathbb{C}^n$ and G acts on V as the matrices act on vectors.
- $G = O_n(\mathbb{C})$ is the orthogonal group over $\mathbb{C}$ formed by matrices g satisfying $g^T g = I$, $V = \mathbb{C}^n$ and G acts on V as the matrices act on vectors.
- $G = GL_n(\mathbb{C})$, $V = M_n(\mathbb{C})$ is the space of $n \times n$ matrices with entries in C and the action of G on V is defined by conjugation.
- $G = O_n(\mathbb{C})$, $V = M_n(\mathbb{C})$ and the action of G on V is defined by conjugation.

We will present in detail our study of these four cases in Section 3. The first and second cases are quite simple.

Proposition 1.1. *There is only one orbit for* $GL_n(\mathbb{C})$ *acting on* $\mathbb{P}(\mathbb{C}^n)$.

Proposition 1.2. *There are two orbits for $O_n(\mathbb{C})$ acting on $\mathbb{P}(\mathbb{C}^n)$. One is formed by the points $x \in \mathbb{P}(\mathbb{C}^n)$ satisfying $x^T \cdot x \neq 0$ and the other is formed by the points $x \in \mathbb{P}(\mathbb{C}^n)$ satisfying $x^T \cdot x = 0$. The two orbits are not isomorphic.*

In the third case, we get the following result.

Theorem 1.3. *Let $V = M_n(\mathbb{C})$ be the representation of $G = GL_n(\mathbb{C})$ given by the conjugation. The notations $G_x, G_{x'}, \Omega_x$ and $\Omega_{x'}$ follow those in Problem 1. Let $x, x' \in \mathbb{P}(V)$ be two points and let $\tilde{x}$ and $\tilde{x}'$ be two matrices representing x and x', respectively. Let $J_{\lambda_1,k_1}, \dots, J_{\lambda_r,k_r}$ (resp. $J_{\lambda'_1,k'_1}, \dots, J_{\lambda'_s,k'_s}$) be Jordan blocks in the Jordan normal form of $\tilde{x}$ (resp. $\tilde{x}'$). Assume that $r = s$ and that up to a reordering we have $k_i = k'_i$ for each i. Suppose there exist $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq -n\beta$ and $\beta \neq 0$ such that*

$$\lambda'_i = \alpha\lambda_i + \beta \sum_{i=1}^{r} k_i\lambda_i \tag{1}$$

for each i. Then G_x and $G_{x'}$ are projectively equivalent, and Ω_x and $\Omega_{x'}$ are projectively equivalent.

In the fourth case, we get the following result.

Theorem 1.4. *Let $V = M_n(\mathbb{C})$ be the representation of $G = O_n(\mathbb{C})$ given by the conjugation. The notations $G_x, G_{x'}, \Omega_x$ and $\Omega_{x'}$ follow those in Problem 1. Let $x, x' \in \mathbb{P}(V)$ be two points and let $\tilde{x}$ and $\tilde{x}'$ be two matrices representing $\tilde{x}$ and $\tilde{x}'$, respectively. If there exist $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying $\alpha \neq \pm\beta$ and $\alpha + \beta \neq -n\gamma$ such that*

$$\tilde{x}' = \alpha\tilde{x} + \beta\tilde{x}^T + \gamma Tr\tilde{x}. \tag{2}$$

Then G_x and $G_{x'}$ are projectively equivalent, and Ω_x and $\Omega_{x'}$ are projectively equivalent.

We prove Theorems 1.3 and 1.4 by explicitly finding a projective transformation that sends one orbit to the other. Representation theory shows that Theorems 1.3 and 1.4 are the best that we can get by using our method. See Lemma 3.7 and Lemma 3.9 for detailed discussions on this topic.

In our study, we also obtain some interesting geometric results using the machinery we develop in the article. They are results of the following form.

Let $\lambda_1, \lambda_2 \in \mathbb{C}$. We define a projective variety Y_{λ_1,λ_2} in $\mathbb{P}^8$ of points with homogeneous coordinates $[y_{11}: y_{12}: y_{13}: y_{21}: y_{22}: y_{23}: y_{31}: y_{32}: y_{33}]$ satisfying the following system of equations:

$$\begin{cases} (2\lambda_1 + \lambda_2)^3(y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11}) \\ \qquad = \lambda_1^2\lambda_2(y_{11} + y_{22} + y_{33})^3 \\ (2\lambda_1 + \lambda_2)^2(y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{33} - y_{22}y_{33}) \\ \qquad = -(2\lambda_1\lambda_2 + \lambda_1^2)(y_{11} + y_{22} + y_{33})^3 \end{cases} \tag{3}$$

Proposition 1.5. *Given two pairs of distinct complex numbers (λ_1, λ_2) and (λ'_1, λ'_2), suppose that*

• *either (λ_1, λ_2) is proportional to (λ'_1, λ'_2) up to a nonzero constant,*

• *or the following conditions hold*

$$\begin{cases} \lambda_2 \neq -2\lambda_1 \\ (2\lambda_1 + \lambda_2)\lambda'_1 \neq \lambda_2\lambda'_2 \\ 2(2\lambda_1 + \lambda_2)\lambda'_1 \neq (3\lambda_1 - \lambda_2)\lambda'_2 \end{cases} \tag{4}$$

Then the projective varieties Y_{λ_1,λ_2} and $Y_{\lambda'_1,\lambda'_2}$ are projectively equivalent in $\mathbb{P}^8$.

This article is organized as follows. In Section 2, we present classical results in representation theory, general topology and projective geometry that will be used in the subsequent sections. In Section 3, we study Problem 1 and prove Propositions 1.1, 1.2 and Theorems 1.3, 1.4. In Section 4, we explore geometric applications of the machinery we developed and prove results in the form of Proposition 1.5.

2. Basic knowledge

In this section, we present some basic knowledge that is necessary to the subsequent sections of the article.

2.1. Representation theory

The material presented here is well-known to experts and we refer the readers to [3] for more details about representation theory. Let k be a field.

Definition 2.1. Let G be a group. A representation of G is a pair (V, ρ) where V is a k-vector space and $\rho: G \to GL(V)$ is a map satisfying: (i) $\rho(e) = IdV$ (ii) $\rho(gh) = \rho(g)\rho(h)$.

Definition 2.2. Let (V, ρ) be a representation of G. A subrepresentation (W, ρ') is a representation of $_G$, with $W \subset V$ a sub-vector space of V and $\rho'(g) = \rho(g)$ for any $g \in G$.

Definition 2.3. Let $(W, \rho') \subset (V, \rho)$ be a subrepresentation. The quotient representation $(V/W, \overline{\rho})$ is a representation of G such that $\overline{\rho}(g) = \overline{\rho(g)}$ where $\overline{\rho(g)} = \overline{\rho(g)(x)}$ for any $\overline{x} \in V/W$.

Definition 2.4. A subrepresentation W of representation V is called trivial if $W = 0$ or $W = V$. A representation V of G is called irreducible if it has no nontrivial subrepresentations.

Definition 2.5. A map $\phi: W \to V$ between representations of a group G is called a morphism (of representations) if: (i) ϕ is k-linear. (ii) $\phi(g.w) = g.\phi(w)$ for all $g \in G, w \in W$.

Lemma 2.6 *(Schur).* (i) *Let W and V be irreducible representations of a group G and $\phi : W \to V$ be a morphism of representations. Then either $\phi = 0$ or ϕ is an isomorphism.*

(ii) *Let k be algebraically closed. Let V be an irreducible finite-dimensional representation of a group G. Let $\phi: V \to V$ be an endomorphism of the representation V. Then there exists $\lambda \in \mathbb{C}$ for which $= \phi\lambda Id_V$.*

Proof. We begin by proving (i). Let $w \in ker\phi$ and $g \in G$. Then since $\phi(gw) = g\phi(w) = 0$, $gw \in ker\phi$. Let $v \in Im\,\phi$ and write $v = \phi(w_0)$. Then, $gv = g\phi(w) = \phi(gw) \in Im\,\phi$. Hence $ker\,\phi \subset W$ and $Im\,\phi \subset V$ are subrepresentations. Either $ker\,\phi = \{0\}$ or $ker\,\phi = W$. If $ker\,\phi = W$, $\phi = 0$. If $ker\,\phi = \{0\}$, ϕ is injective. In addition, $Im\,\phi \subset V \Longrightarrow Im\,\phi = \{0\} *= V$, meaning that ϕ is surjective and thus bijective. It is not hard to verify that $\phi^{-1}: W \to V$ is also a morphism of representations. We conclude that $\phi: W \to V$ is an isomorphism, as desired.

Next, we prove (ii). By linear algebra, there exists a nonzero vector $v \in V$ and a scalar $\lambda \in \mathbb{C}$ such that $\phi(v) = \lambda v$. The linear map $\phi - \lambda Id_V : V \to V$ is a morphism of representations. By (i), either $\phi - \lambda Id_V = 0$ or $\phi - \lambda Id_V$ is an isomorphism. However, since there exists nonzero $v \in V$ for which $\phi(v) = \lambda v$, $\phi - \lambda Id_V$ is not injective. Thus $\phi = \lambda Id_V$.

2.2. General topology

In this part, we present some topological language from the perspective of metric spaces. One may refer to [4] for a more abstract approach. We include this part to provide a basis for our usage of Euclidean topology in complex projective geometry.

Recall how continuity of maps is defined in differential calculus.

Definition 2.7. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$, there exist $\delta > 0$ such that if $x \in \mathbb{R}$ satisfies $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}$. The Euclidean definition of distance is $dist(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$. Note that we can also use a non-Euclidean definition of "distance", such as $dist_p(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p}$ for $p \geq 1$.

Definition 2.8. A map $f : \mathbb{R} \to \mathbb{R}$ is called continuous at $x_0 \in \mathbb{R}^n$ if for all $\epsilon > 0$, there exist $\delta > 0$ such that if $x \in \mathbb{R}^n$ satisfies $dist(x, x_0) < \delta$, then $dist(f(x), f(x_0)) < \epsilon$.

Motivated by this, we introduce the concept of metric spaces to talk about continuity in a more general setting.

Definition 2.9. A metric space is a set X endowed with a function
$$d: X \times X \to \mathbb{R}, \tag{5}$$
called the metric function (or distance function), satisfying the following properties:

(i) $d(x, y) = 0$ iff $x = y$,
(ii) $d(x, y) = d(y, x), and$
(iii) $d(x, z) \leq d(x, y) + d(y, z)$

Definition 2.10. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f: X \to Y$ is called continuous at $x_0 \in X$ if for all $\epsilon > 0$, there exist $\delta > 0$ such that if $x \in X$ and $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \epsilon$. Such a map is called continuous if it is continuous at each $x_0 \in X$.

Open and closed subsets in metric spaces

Let (X, d_X) be a metric space.

Definition 2.11. An open ball centered at $x_0 \in X$ with radius $r > 0$ is the set $B(x_0, r) := \{x \in X: d_X(x, x_0) < r\}$. A closed ball centered at $x_0 \in X$ with radius $r > 0$ is the set $\overline{B}(x_0, r) := \{x \in X: d_X(x, x_0) \leq r\}$.

Definition 2.12. (i) A subset $U \subset X$ is called open if for any $x_0 \in U$, there exists $r > 0$ such that $B(x_0, r) \subset U$.

(ii) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in X. The sequence is called convergent in X if there is a point $A \in X$ such that for all $\epsilon > 0$, there exist $N \in \mathbb{N}$ for which $d_X(x_n, A) < \epsilon$ for all $n > N$.

(iii) A subset $F \subset X$ is closed if for any convergent sequence $\{x_n\}_{n=1}^{\infty} \subset F$, the convergence point lies in F.

Proposition 2.13. *Let Y be a subset of X. Then $Y \subset X$ is open if and only if $Y^c \subset X$ is closed.*

Proof. Suppose $Y \subset X$ is open. Let the sequence $\{x_n\}_{n=1}^{\infty} \subset Y^c$ converge to $A \in X$. That is, for any, there exists $N \in \mathbb{N}$ such that for all $n > N$. Assume for the sake of contradiction that $A \in Y$. Since Y is open, there exists $r > 0$ such that $B(A, r) \subset Y$. Pick $\epsilon = r$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in B(A, r) \subset Y$, contradicting $x_n \in Y^c$. Hence $A \in Y^c$ for any choice of $\{x_n\}_{n=1}^{\infty}$ and so $Y^c \subset X$ is closed.

Suppose $Y^c \subset X$ is closed. Let y be an arbitrary point in Y. Assume for the sake of contradiction that for each $r > 0$, $B(y, r) \cap Y^c \neq \phi$. Picking $r = 1/n$, we can construct a sequence $\{x_n\}_{n=1}^{\infty} \subset Y^c$ such that $x_n \in B(y, 1/n) \cap Y^c$. This sequence converges to $y \in X$. Since Y^c is closed, $y \in Y^c$, contradicting $y \in Y$. Thus there must exist $r > 0$ so that $B(y, r) \subset Y^c$. Hence $Y \subset X$ is open.

Theorem 2.14. *Let X and Y be metric spaces. Let $f : X \to Y$ be a map. Then the following three statements are equivalent:*

(i)   *f is continuous.*

(ii)   *for any open subset $V \subset Y$, the preimage $f^{-1}(V) \subset X$ is open.*

(iii)   *for any closed subset $W \subset Y$, the preimage $f^{-1}(W) \subset X$ is closed.*

Proof. By Proposition 2.13, (ii) and (iii) are equivalent. We will prove that (i) if and only if (ii).

Suppose f is continuous. Let $V \subset Y$ be an open subset. Let $x \in f^{-1}(V)$ be an arbitrary point. Since $f(x) \in V$ and $V \subset Y$ is open, there exist $\epsilon > 0$ such that $B(f(x), \epsilon) \subset V$. By the continuity of f, there exists $\delta > 0$ for which $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Then $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon)) \subset f^{-1}(V)$ and hence $f^{-1}(V)$ is open.

Suppose for any open subset $V \subset Y$, the preimage $f^{-1}(V) \subset X$ is open. Let x be an arbitrary point in X. For any $\epsilon > 0$, consider the open ball $B(f(x), \epsilon) \subset Y$, which is an open subset of Y. Then $f^{-1}(B(f(x), \epsilon)) \subset X$ is open. Since clearly $x \in f^{-1}(B(f(x), \epsilon))$, there is $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$. Then for all $y \in X$ satisfying $d_X(y, x) < \delta$, we have $d_Y(f(y), f(x)) < \epsilon$. Hence f is continuous at x. Therefore f is continuous.

As we can see from Theorem 2.14, the notion of continuity can be talked about without mentioning metrics. All we need to be able to talk about continuity is the concept of open subsets (or equivalently the concept of closed subsets). This is the motivation for mathematicians to define topological spaces.

Definition 2.15. A topological space is a set X endowed with a set τ of subsets of X satisfying the following properties:

(i) $\phi, X \in \tau$;

(ii)   Let $\{U_i\}_{i \in I} \subset \tau$. Then $\cup_{i \in I} U_i \in \tau$;

(iii)   Let $U, V \in \tau$. Then $U \cap V \in \tau$.

The set τ is called a topology of X and its elements are called open subsets of X.

Proposition 2.16. *Let X be a metric space. Let τ be the set of open subsets of X defined by the metric. Then (X, τ) is a topological space.*

Proof. First, $\varphi, X \in \tau$ is clearly true.

Second, let $\{U_i\}_{i \in I}$ be a family of open subsets of X. Let $x \in \cup_{i \in I} U_i$. So there exists $i_0 \in I$ such that $x \in U_{i_0}$. But since $U_{i_0} \subset X$ is open, there exists $r > 0$ such that $B(x, r) \subset U_{i_0} \subset \cup_{i \in I} U_i$. Thus, $\cup_{i \in I} U_i$ is open, as desired.

Third, let U, V be open subsets of X. We must show that $U \cap V$ is open. Let $x \in U \cap V$. Then there exists $r_U, r_V > 0$ such that $B(x, r_U) \subset U$ and $B(x, r_V) \subset V$. Pick $r = \min\{r_U, r_V\} > 0$ and we have $B(x, r) \subset U \cap V$. Hence $U \cap V$ is an open subset, as desired.

Hence, we may say that metric spacs induce topological spaces. Motivated by Theorem 2.14, we define the concept of continuity for topological spaces.

Definition 2.17. Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let $f : X \to Y$ be a map. The map f is continuous if for any $V \in \tau_Y$, $f^{-1}(V) \in \tau_X$.

Now we introduce two notions that are important in this article.

Definition 2.18. Let X be a topological space. Let Y be a subset of X.

(i) The closure of Y, denoted by $\bar{Y}$, is the smallest closed subset of X containing Y.

(ii) The interior of Y, denoted by Y°, is the largest open subset of X contained in Y.

2.3. Projective geometry

We present here some basic notions in projective geometry, which we will use later. Let k be a field.

Definition 2.19. Let V be a k-vector space. Denote $\mathbb{P}(V)$ to be the set of 1-dimensional subspaces in V. When $V = k^n$, we write $\mathbb{P}(V) = \mathbb{P}^{n-1}(k)$.

Definition 2.20. The natural map $\pi : V - \{0\} \to \mathbb{P}(V)$ takes x to $[x]$, the 1-dimensional subspace of V generated by x.

Definition 2.21. A projective transformation $\phi : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ is given by an invertible $(n + 1) \times (n + 1)$ matrix A such that $\phi([x]) = [Ax]$.

Definition 2.22. Let X and Y be subsets of $\mathbb{P}^n(k)$. X and Y are called projectively equivalent if there exists a projective transformation that takes X to Y.

When $k = \mathbb{C}$, $V - \{0\}$ has a Euclidean topology. Using this topology, we can define a topology on $\mathbb{P}(V)$.

Definition 2.23. Define a subset $Y \subset P(V)$ to be open if and only if $\pi^{-1}(Y) \subset V - \{0\}$ is open. Let τ be the set of open subsets of $\mathbb{P}(V)$ defined this way.

One checks readily that τ is a topology on $\mathbb{P}(V)$.

Proposition 2.24. *Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial. Define*

$$V(f) := \{[x_0 : x_1 : \cdots : x_n] : f(x_0, \dots, x_n) = 0\} \subset \mathbb{P}^n(\mathbb{C}). \tag{6}$$

Then $V(f)$ is closed with respect to the topology τ defined in Definition 2.23.

Proof. It suffices to prove that $\pi^{-1}(V(f))$ is closed. We define $\tilde{f}$ as a function on $\mathbb{C}^{n+1} - \{0\} \to \mathbb{C}$ as follows:

$$\tilde{f} : (x_0, \dots, x_n) \mapsto f(x_0, \dots, x_n). \tag{7}$$

The map $\tilde{f}$ is continuous because it is a polynomial function. Hence $\pi^{-1}(V(f)) = \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} - \{0\} : \tilde{f}(x_0, \dots, x_n) = 0\} = \tilde{f}^{-1}(\{0\})$ is closed in $\mathbb{C}^{n+1} - \{0\}$.

3. The geometry of orbits

In this section, we study Problem 1 in detail through the following four examples.

- $G = GL_n(\mathbb{C})$ is the general linear group over $\mathbb{C}$, $V = \mathbb{C}^n$ and G acts on V as matrices act on vectors.
- $G = O_n(\mathbb{C})$ is the orthogonal group over $\mathbb{C}$ formed by matrices g satisfying $g^\top g = I$, $V = \mathbb{C}^n$ and G acts on V as matrices act on vectors.
- $G = GL_n(\mathbb{C})$, $V = M_n(\mathbb{C})$ is the space of $n \times n$ matrices with entries in $\mathbb{C}$ and the action of G on V is defined by conjugation.
- $G = O_n(\mathbb{C})$, $V = M_n(\mathbb{C})$ and the action of G on V is defined by conjugation.

3.1. The representation of $GL_n(\mathbb{C})$ on $\mathbb{C}^n$

Lemma 3.1. *Let $v, v' \in \mathbb{C}^n$ be nonzero vectors. There exists $g \in GL_n(\mathbb{C})$ for which $v' = gv$.*

Proof. Let $\{e_1 = v, e_2, \ldots, e_n\}$ be a basis of $\mathbb{C}^n$, and let $w = (1,0,\cdots,0)^\top$. Note that $A = (e_1 \quad \cdots \quad e_n) \in GL_n(\mathbb{C})$ is invertible. In addition, $Aw = v \Rightarrow w = A^{-1}v$. Similarly, we can let $\{e_1' = v', e_2', \ldots, e_n'\}$ be a basis of $\mathbb{C}^n$ and $A' = (e_1' \quad \cdots \quad e_n') \in GL_n(\mathbb{C})$. We then have $A'w = v' \Rightarrow A'A^{-1}v = v'$. Pick $g = A'A^{-1} \in GL_n(\mathbb{C})$.

Let $x, x' \in \mathbb{P}(V)$. Since by Lemma 3.1 any nonzero vector can be expressed as a product of a matrix in $GL_n(\mathbb{C})$ and any other nonzero vector, $G_x = \mathbb{P}(V) = G_{x'}$. Consequently, $\Omega_x \cong \Omega_{x'}$. Since projective transformations are defined by linear maps, they are continuous. Hence, a projective transformation sends limit points to limit points. If two subsets are projectively equivalent, so are their closures.

3.2. The representation of $O_n(\mathbb{C})$ on $\mathbb{C}^n$

Lemma 3.2. *Let $x \in \mathbb{C}^n$ be a vector such that $x^\top \cdot x \neq 0$. Then there is a basis $\{e_1 = x, e_2, \ldots, e_n\}$ such that*

$$\begin{cases} e_i^\top \cdot e_j = 0 \text{ "for" } i \neq j \\ \quad e_i^\top \cdot e_i \neq 0. \end{cases} \tag{8}$$

Proof. Proceed by induction on $\dim V$. If $\dim V = 1$, the result is trivial. Assume that the lemma holds for all vector spaces of dimension less than $\dim V$. Pick a vector $x \in V$ for which $x \cdot x = 0$. For any $y \in V$,

$$x \cdot y = \frac{1}{2}\{(x + y) \cdot (x + y) - x \cdot x - y \cdot y\} = 0 \tag{9}$$

Consider $W = \langle x \rangle^\perp = \{v \in V : v \cdot x = 0\}$. Since

$$\begin{cases} \dim W = \dim V - 1 < \dim V \\ \quad W \cap \langle x \rangle = \{0\}, \end{cases} \tag{10}$$

$V = W \oplus \langle x \rangle$. The statement is proven by induction.

Proposition 3.3. *Denote the orbit of x under the action $O_n(\mathbb{C})$ on $\mathbb{P}(\mathbb{C}^n)$ by G_x and the closure of G_x in $\mathbb{P}(V)$ by Ω_x. Let $x, x' \in \mathbb{P}(\mathbb{C}^n)$. If $x^\top \cdot x \neq 0$ and $x'^\top \cdot x' \neq 0$, then $G_x = G_{x'}$.*

Proof. Let $x, x' \in \mathbb{P}(\mathbb{C}^n)$. Pick bases $\{e_1 = x, e_2, \ldots, e_n\}$ and $\{f_1 = x', f_2, \ldots, f_n\}$ as in the lemma. Let $x^\top \cdot x = x'^\top \cdot x' = \lambda$. Since $e_i^\top \cdot e_i \neq 0$ and $f_i^\top \cdot f_i \neq 0$, we may assume that $e_i^\top \cdot e_i = f_i^\top \cdot f_i = \lambda$ for all $i \geq 2$. Take $g = (f_1, f_2, \ldots, f_n)(e_1, e_2, \ldots, e_n)^{-1}$. Note that

$$(e_1, e_2, \ldots, e_n)^\top \cdot (e_1, e_2, \ldots, e_n) = (f_1, f_2, \ldots, f_n)^\top \cdot (f_1, f_2, \ldots, f_n) = \lambda I_n. \tag{11}$$

Hence, $((e_1, e_2, \ldots, e_n)^\top)^{-1} \cdot (e_1, e_2, \ldots, e_n)^{-1} = \lambda^{-1} I_n$.

$$(e_1, e_2, \ldots, e_n)^\top g^\top \cdot g(e_1, e_2, \ldots, e_n) = (f_1, f_2, \ldots, f_n)^\top (f_1, f_2, \ldots, f_n) = \lambda I_n. \tag{12}$$

Since

$$g^\top g = \lambda((e_1, e_2, \ldots, e_n)^\top)^{-1} \cdot (e_1, e_2, \ldots, e_n)^{-1} = \lambda \lambda^{-1} I_n = I_n, \tag{13}$$

$g \in G$. Thus x' is in the orbit of x. Therefore $G_x \cong \mathbb{P}(V) \cong G_{x'}$ and consequently $\Omega_x \cong \Omega_{x'}$.

Proposition 3.4. *Denote the orbit of x under the action $O_n(\mathbb{C})$ on $\mathbb{P}(\mathbb{C}^n)$ by G_x and the closure of G_x in $\mathbb{P}(V)$ by Ω_x. Let $x, x' \in \mathbb{P}(\mathbb{C}^n)$. If $x^\top \cdot x = x'^\top \cdot x' = 0$, then $G_x = G_{x'}$.*

Proof. We begin by introducing the following claim.

Claim. Let nonzero $x \in V$ satisfy $x^\top \cdot x = 0$. Then there exists a 2-dimensional subspace $W \subset V$ containing x and $V = W \oplus W^\perp$. Furthermore, the symmetric bilinear form on $V \to V$ taking $(x, y) \mapsto x^\top \cdot y$ is nondegenerate.

Proof. Since $x^\top \cdot x = 0$ and $(x, y) \mapsto x^\top \cdot y$ is nondegenerate, we can find $y \neq x \in V$ such that $x^\top \cdot y \neq 0$. $W = \langle x, y \rangle$ is a 2-dimensional subspace of V. Let $v \in W$ so that $v^\top \cdot w = 0$ for all $w \in W$. Then $v = 0$, so the bilinear form is nondegenerate on W. Similarly, the bilinear form is nondegenerate on $W^\perp$.

Let $v \in W \cap W^\perp$. Since $v \in W$, express it as $v = ax + by$ for $a, b \in \mathbb{C}$. Since $v \in W^\perp$, $v^\top \cdot x = v^\top \cdot y = 0$. Hence,

$$\begin{cases} b(y^\top \cdot x) = 0 \\ a(x^\top \cdot y) + b(y^\top \cdot y) = 0. \end{cases} \tag{14}$$

Since $x^\top \cdot y = y^\top \cdot x \neq 0$, $a = b = 0$. Hence, $v = 0$. $W \cap W^\perp = \{0\}$ and so $V = W \oplus W^\perp$. The claim is proven.

By the claim,

$$V = W_x \oplus W_x^\perp = W_{x'} \oplus W_{x'}^\perp.$$

By Lemma 3.1, $W_x^\perp \overset{\sim}{\to} W_{x'}^\perp$ via an orthogonal matrix g'. Since W_x and $W_{x'}$ are 2-dimensional vector spaces with induced bilinear forms, we can find another orthogonal matrix g'' taking $W_x \to W_{x'}$ and sending $x \mapsto x'$. Then $g = g'g'' \in O_n(\mathbb{C})$ sends $x \mapsto x'$, and thus there is an orthogonal matrix $g \in O_n(\mathbb{C})$ such that $x' = gx$. It follows that $G_x \cong G_{x'}$.

Proposition 3.5. *The two orbits mentioned in Propositions 3.3 and 3.4 are not isomorphic.*
Proof. This is due to dimension reasons. The orbit in Proposition 3.3 has dimension $n - 1$ whereas the orbit in Proposition 3.4 has dimension $n - 2$.

As a corollary, all of these propositions are true for the closures as well.

3.3. The representation of $GL_n(\mathbb{C})$ on $M_n(\mathbb{C})$

In this part, we prove Theorem 1.3 and discuss how can we use our method to get more general results.
Proposition 3.6. *Here we use the assumptions on α and β as in Theorem 1.3. The linear map $\phi: \mathbb{P}M_n(\mathbb{C}) \to \mathbb{P}M_n(\mathbb{C})$ of the form $\phi: A \mapsto \alpha A + \beta \mathrm{Tr}(A)$ is a projective transformation that sends G_x to $G_{x'}$.*
Proof. The first $k_1 \times k_1$ submatrix of $\phi(x)$ can be expressed as

$$\begin{pmatrix} \lambda_1' & \alpha & & \\ & \ddots & \ddots & \\ & & \ddots & \alpha \\ & & & \lambda_1' \end{pmatrix}, \tag{15}$$

while the first $k_1 \times k_1$ submatrix of x' can be expressed as

$$\begin{pmatrix} \lambda_1' & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1' \end{pmatrix}. \tag{16}$$

Let A be the following matrix:

$$\begin{pmatrix} 1 & & & & \\ & \frac{1}{\alpha} & & & \\ & & \frac{1}{\alpha^2} & & \\ & & & \ddots & \\ & & & & \frac{1}{\alpha^{k_i-1}} \end{pmatrix}. \tag{17}$$

We can easily verify that x' and $\phi(x)$ are conjugates by comparing each of these blocks:

$$\begin{pmatrix} \lambda_1' & \alpha & & \\ & \ddots & \ddots & \\ & & \ddots & \alpha \\ & & & \lambda_1' \end{pmatrix} = A \begin{pmatrix} \lambda_1' & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1' \end{pmatrix} = A^{-1}. \tag{18}$$

Therefore, the map ϕ sends G_x to $G_{x'}$.

Let $\phi'(A) = \alpha^{-1}A - \frac{\alpha^{-1}\beta}{\alpha+n\beta}\mathrm{Tr}(A)$ be another linear map on $\mathbb{P}M_n(\mathbb{C}) \to \mathbb{P}M_n(\mathbb{C})$. We can verify that ϕ' is the inverse of ϕ. Since ϕ has an inverse, it is a projective transformation, as desired.
Proof of Theorem 1.3. The proposition above proves that G_x is projectively equivalent to $G_{x'}$. Since the projective transformation $\phi: \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$ is continuous, it sends limit points to limit points. Hence, the closures Ω_x and $\Omega_{x'}$ are projectively equivalent by the same projective transformation.

Remark. In the proof of Theorem 1.3, we tried to find a linear map

$$\phi \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \tag{19}$$

satisfying

- $\phi(gAg^{-1}) = g\phi(A)g^{-1}$ for each $A \in M_n(\mathbb{C})$ and for each $g \in GL_n(\mathbb{C})$,
- $\phi(x)$ is conjugate to x', and
- ϕ is invertible.

Our construction was $\phi \colon A \mapsto \alpha A + \beta \mathrm{Tr}(A)$ for $\alpha \neq 0$ and $\alpha \neq -n\beta$. In fact, we can see that if ϕ is satisfies the three conditions listed above, then ϕ must be of the form $\phi \colon A \mapsto \alpha A + \beta \mathrm{Tr}(A)$ for $\alpha \neq 0$ and $\alpha \neq -n\beta$ (Lemma 3.7). For G_x and $G_{x'}$ to be projectively equivalent, the first condition in the list above is more restrictive than necessary. One only needs to ask

$$\forall A \in M_n(\mathbb{C}), \forall g \in GL_n(\mathbb{C}), \exists \lambda \in \mathbb{C}^\times \text{ such that } \phi(gAg^{-1}) = \lambda g\phi(A)g^{-1}. \tag{20}$$

It would be interesting to explore whether the looser condition can give us a generalization of Theorem 1.3.

Lemma 3.7. *Let $\phi \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map such that for all $g \in GL_n(\mathbb{C})$ and for all $A \in M_n(\mathbb{C})$, $\phi(gAg^{-1}) = g\phi(A)g^{-1}$. Then there exists $\lambda_1, \lambda_2 \in \mathbb{C}$ for which $\lambda_1 \neq 0$, $n\lambda_2 \neq -\lambda_1$, and $\phi(A) = \lambda_1 A + \lambda_2 \mathrm{Tr}(A)$.*

Proof. We view $M_n(\mathbb{C})$ as a representation of $GL_n(\mathbb{C})$ by

$$g.x := gxg^{-1}. \tag{21}$$

Note that this representation is not irreducible. We can write $M_n(\mathbb{C}) = M_n^0(\mathbb{C}) \oplus \mathbb{C}I_n$, where $M_n^0(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \colon \mathrm{Tr}(A) = 0\}$. [5, Chapter V] contains a proof of the following classical result.

Result. $M_n^0(\mathbb{C})$ and $\mathbb{C}I_n$ are irreducible representations of $GL_n(\mathbb{C})$.

Let $\phi \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a morphism of representations $\phi = \phi_{00} + \phi_{01} + \phi_{10} + \phi_{11}$ satisfying the conditions of the Lemma, where

$$\begin{cases} \phi_{00} \colon M_n^0(\mathbb{C}) \to M_n^0(\mathbb{C}) \\ \phi_{01} \colon M_n^0(\mathbb{C}) \to \mathbb{C}I_n \\ \phi_{10} \colon \mathbb{C}I_n \to M_n^0(\mathbb{C}) \\ \phi_{11} \colon \mathbb{C}I_n \to \mathbb{C}I_n. \end{cases} \tag{22}$$

Hence $\phi_{00} = \alpha I_n$ and $\phi_{11} = \beta I_n$, whereas $\phi_{01} = \phi_{10} = 0$, by Schur's Lemma. Since ϕ must be invertible, $\alpha \neq 0$ and $\beta \neq 0$, and

$$\phi = \alpha \mathrm{Id}_{M_n^0(\mathbb{C})} + \beta \mathrm{Id}_{\mathbb{C}I_n}. \tag{23}$$

Let $A \in M_n(\mathbb{C})$. We can write $A = (A - \frac{1}{n}\mathrm{Tr}(A)) + (\frac{1}{n}\mathrm{Tr}(A))$. Then,

$$\phi(A) = \alpha\left(A - \frac{1}{n}\mathrm{Tr}(A)\right) + \beta\left(\frac{1}{n}\mathrm{Tr}(A)\right)$$
$$= \alpha A + \frac{1}{n}(\beta - \alpha)\mathrm{Tr}(A)$$
$$= \lambda_1 A + \lambda_2 \mathrm{Tr}(A).$$

Hence we have our desired values for λ_1 and λ_2, and the lemma is proven.

3.4. The representation of $O_n(\mathbb{C})$ on $M_n(\mathbb{C})$

Proposition 3.8. *If there exists $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha \neq \pm\beta$, $\alpha + \beta \neq -n\gamma$, and $y = \alpha x + \beta x^\top + \gamma \mathrm{Tr}(x)$, then the linear map $\phi \colon \mathbb{P}M_n(\mathbb{C}) \to \mathbb{P}M_n(\mathbb{C})$ of the form $\phi \colon A \mapsto \alpha A + \beta A^\top + \gamma \mathrm{Tr}(A)$ is a projective transformation that sends G_x to G_y.*

Proof. Let us check that $\phi(gAg^{-1}) = g\phi(A)g^{-1}$ for all orthogonal matrices g. Notice that $g^\top = g^{-1}$ for g an orthogonal matrix. Hence,

$$\phi(gAg^{-1}) = \alpha(gAg^\top) + \beta(gAg^\top)^\top + \gamma \mathrm{Tr}(gAg^{-1})$$
$$= \alpha(gAg^\top) + \beta(gA^\top g^\top) + \gamma \mathrm{Tr}(A)$$
$$= g(\alpha A + \beta A^\top + \gamma \mathrm{Tr}(A))g^\top$$
$$= g\phi(A)g^{-1}.$$

It follows that ϕ does indeed send G_x to G_y.

Now we want to show that ϕ is invertible. Let be another linear map on $\mathbb{P}M_n(\mathbb{C}) \to \mathbb{P}M_n(\mathbb{C})$. We can verify that ϕ' is the inverse of ϕ. Since ϕ has an inverse, it is a projective transformation, as desired.

This proves Theorem 1.4.

Lemma 3.9. *Let $n \geq 3$. Let $\phi: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map such that for all $g \in O_n(\mathbb{C})$ and for all $A \in M_n(\mathbb{C})$, $\phi(gAg^{-1}) = g\phi(A)g^{-1}$. Then there exists $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying the conditions given in Proposition 3.8 and such that $\phi(A) = \alpha A + \beta A^{\mathsf{T}} + \gamma Tr(A)$.*

Proof. The representation V of G can be expressed as
$$V = \mathrm{Sym}^\circ \oplus \mathbb{C}I_n \oplus \mathrm{Ant} = V_1 \oplus V_2 \oplus V_3, \tag{24}$$
where $\mathrm{Sym}^\circ = \{y \in M_n(\mathbb{C}): y^{\mathsf{T}} = y, \mathrm{Tr}(y) = 0\}$ and $\mathrm{Ant} = \{y \in M_n(\mathbb{C}): y^{\mathsf{T}} = -y\}$. The representations Sym, $\mathbb{C}I_n$, and Ant are non-isomorphic irreducible representations of $G = O_n(\mathbb{C})$ [5, Chapter V]. Let $\phi: V \to V$ be an isomorphism on V. Then let $\phi = \Sigma_{i,j}\phi_{ij}$ where $\phi_{ij}: V_i \to V_j$.

By Schur's Lemma, $\phi = \lambda_1 \mathrm{Id}_{V_1} + \lambda_2 \mathrm{Id}_{V_2} + \lambda_3 \mathrm{Id}_{V_3}$ for nonzero λ_1, λ_2, and λ_3. For any $x \in M_n(\mathbb{C})$,
$$x = \frac{x + x^{\mathsf{T}}}{2} + \frac{x - x^{\mathsf{T}}}{2} = \left[\frac{x + x^{\mathsf{T}}}{2} - \frac{1}{n}\mathrm{Tr}(x)\right] + \left[\frac{1}{n}\mathrm{Tr}(x)\right] + \left[\frac{x - x^{\mathsf{T}}}{2}\right].$$
$$\underbrace{\phantom{\left[\frac{x + x^{\mathsf{T}}}{2} - \frac{1}{n}\mathrm{Tr}(x)\right]}}_{\in \mathrm{Sym}^\circ} \quad \underbrace{\phantom{\left[\frac{1}{n}\mathrm{Tr}(x)\right]}}_{\in \mathbb{C}I_n} \quad \underbrace{\phantom{\left[\frac{x - x^{\mathsf{T}}}{2}\right]}}_{\in \mathrm{Ant}}$$
$$\phi(x) = \lambda_1\left[\frac{x + x^{\mathsf{T}}}{2} - \frac{1}{n}\mathrm{Tr}(x)\right] + \lambda_2\left[\frac{1}{n}\mathrm{Tr}(x)\right] + \lambda_3\left[\frac{x - x^{\mathsf{T}}}{2}\right].$$

We may pick $\alpha = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_3$, $\beta = \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_3$, $\gamma = \frac{1}{n}(\lambda_1 - \lambda_2)$, which satisfy the required conditions.

4. Geometric applications

In this section, we exploit Theorem 1.3 to give some algebraic geometry results. To do this, we need to introduce a linear algebra lemma (Lemma 4.1).

4.1. A linear algebra lemma

Let x be an element in $\mathbb{P}M_n(\mathbb{C})$, with
$$x \sim \begin{pmatrix} J_{\lambda_1, k_1} & & \\ & \ddots & \\ & & J_{\lambda_r, k_r} \end{pmatrix}. \tag{25}$$

Lemma 4.1. *Let $\lambda_1, \ldots, \lambda_r$ be distinct. The closure Ω_x of the orbit of x under the conjugation action of the group $GL_n(\mathbb{C})$ is the set of classes of matrices $y \in M_n(\mathbb{C})$ for which there exists $x_0 \in \mathbb{C}^\times$ satisfying the following condition:*
$$det(y - T \cdot I_n) = \prod_{i=1}^{r} (x_0\lambda_i - T)^{k_i}. \tag{26}$$

Proof. Let Θ_x be the set of classes of matrices $y \in M_n(\mathbb{C})$ for which there exists $x_0 \in \mathbb{C}^\times$ satisfying the following condition:
$$det(y - T \cdot I_n) = \prod_{i=1}^{r} (x_0\lambda_i - T)^{k_i}. \tag{27}$$

To show that Θ_x is the closure of the orbit G_x of x, we need to prove that Θ_x is closed in $\mathbb{P}M_n(\mathbb{C})$ and that every element in Θ_x can be approximated by a sequence in G_x.

Let us first prove the closedness of Θ_x. Each side of the equation (26) is a polynomial in T. On the left-hand-side, the coefficients of this polynomials are homogeneous polynomials in the entries of the matrix y, whereas on the right-hand-side the coefficients are scalars expressed by x_0 and λ_i Comparing the coefficients of the two sides, we get $n + 1$ equations that y must satisfy. Since x_0 is undetermined, we wish to eliminate this variable. By doing so, we get n homogeneous polynomials that characterize elements in Θ_x. Hence, Θ_x is the zero locus in $\mathbb{P}M_n(\mathbb{C})$ of n homogeneous polynomials. By Proposition 2.24, Θ_x is an intersection of closed subsets in $\mathbb{P}M_n(\mathbb{C})$ and is thus closed.

Next, we show that every element in Θ_x can be approximated by a sequence in G_x. To this end, we first prove a

Claim. Θ_x contains exactly some $x' \in \mathbb{P}M_n(\mathbb{C})$ such that if we regard x' as a representing matrix, x' is similar to a matrix of the form

$$
\begin{pmatrix}
x_0 J_{\lambda_1, k_{11}} & & & & & \\
& x_0 J_{\lambda_1, k_{12}} & & & & \\
& & \ddots & & & \\
& & & x_0 J_{\lambda_1, k_{1l_1}} & & \\
& & & & \ddots & \\
& & & & & x_0 J_{\lambda_r, k_{rl_r}}
\end{pmatrix}, \tag{28}
$$

where $x_0 \in \mathbb{C}^\times$ and $k_{i1} + \cdots + k_{ir_i} = k_i$ for every $i = 1, 2, \ldots, r$.

Proof. Let $\mathcal{J}_{x'}$ denote the matrix in (28). Suppose that x' satisfies the condition in the Claim. Since x' and $\mathcal{J}_{x'}$ are similar, we have $\mathcal{J}_{x'} = Qx'Q^{-1}$ for some $Q \in GL_n(\mathbb{C})$. Suppose $x' \in \Theta_x$. Then

$$
\begin{aligned}
\det(x' - T \cdot I_n) &= \det\left(Q(x' - T \cdot I_n)Q^{-1}\right) \\
&= \det(\mathcal{J}_{x'} - T \cdot I_n) \\
&= \prod_{i=1}^{r} \prod_{j=1}^{l_i} \det\left(x_0 J_{\lambda_i, k_{ij}} - T \cdot I_{k_{ij}}\right) \\
&= \prod_{i=1}^{r} (x_0 \lambda_i - T)^{k_i}.
\end{aligned}
$$

Thus $x' \in \Theta_x$ by the definition of Θ_x. Now we consider the inverse direction. Suppose that $x' \in \Theta_x$. Since x' satisfies $\det(x' - T \cdot I_n) = \prod_{i=1}^{r} (x_0 \lambda_i - T)^{k_i}$ for some $x_0 \in \mathbb{C}^\times$, the Jordan normal form of x' should be

$$
J_{x'} = \begin{pmatrix}
J_{x_0 \lambda_1, k_{11}} & & & & & \\
& J_{x_0 \lambda_1, k_{12}} & & & & \\
& & \ddots & & & \\
& & & J_{x_0 \lambda_1, k_{1l_1}} & & \\
& & & & \ddots & \\
& & & & & J_{x_0 \lambda_r, k_{rl_r}}
\end{pmatrix}, \tag{29}
$$

However, $J_{x'}$ and $\mathcal{J}_{x'}$ are similar by Lemma 4.2 below.

Lemma 4.2. *Let $a_1, \ldots, a_{d-1}, b_1, \ldots, b_{d-1}$ be nonzero complex numbers. Let λ be an arbitrary complex number. Then the matrices*

$$
A = \begin{pmatrix}
\lambda & a_1 & & & & \\
& \lambda & a_2 & & & \\
& & \ddots & \ddots & & \\
& & & \lambda & a_i & \\
& & & & \ddots & \ddots \\
& & & & & \lambda
\end{pmatrix} \tag{30}
$$

and

$$
B = \begin{pmatrix}
\lambda & b_1 & & & & \\
& \lambda & b_2 & & & \\
& & \ddots & \ddots & & \\
& & & \lambda & b_i & \\
& & & & \ddots & \ddots \\
& & & & & \lambda
\end{pmatrix} \tag{31}
$$

are similar.

Proof. We will define the sequence $\{\alpha_k\}_{k=1,\dots,d}$ of nonzero complex numbers as follows. Set $\alpha_1 = 1$. For $k > 1$, inductively set $\alpha_k = \frac{a_1 \dots a_{k-1}}{b_1 \dots b_{k-1}}$. Let $Q = \operatorname{diag}(\alpha_1, \dots, \alpha_d)$. Since each α_k is nonzero, the matrix Q is invertible. We can easily check that $B = QAQ^{-1}$.

This lemma concludes our proof of the claim.

Now let us return to the proof of Lemma 4.1. Let $y \in \Theta$. We want to show that y can be approximated by a sequence in G_x. By the claim, we may assume that the Jordan normal form of y, up to scaling, is

$$J_y = \begin{pmatrix} J_{\lambda_1,k_{11}} & & & & & \\ & J_{\lambda_1,k_{12}} & & & & \\ & & \ddots & & & \\ & & & J_{\lambda_1,k_{1l_1}} & & \\ & & & & \ddots & \\ & & & & & J_{\lambda_r,k_{rl_r}} \end{pmatrix}, \tag{32}$$

The first $k_1 \times k_1$ submatrix of J_y is:

$$A = \begin{pmatrix} J_{\lambda_1,k_{11}} & & & \\ & J_{\lambda_1,k_{12}} & & \\ & & \ddots & \\ & & & J_{\lambda_1,k_{1l_1}} \end{pmatrix}, \tag{33}$$

which can be approximated by the sequence $\{A_n\}$ define by

$$A_n = \begin{pmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & \lambda & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \lambda & 1/n \\ & & & & & & \lambda & 1 \\ & & & & & & & \ddots & \ddots \end{pmatrix} \tag{34}$$

where each A_n is similar to J_{λ_1,k_1} by Lemma 4.2. For each $k_i \times k_i$ submatrix of J_y, we may do the same process, and thus we can conclude that J_y can be approximated by a sequence in G_x. Since y is similar to J_y, y can also be approximated by a sequence in G_x, as desired. Hence the proof of Lemma 4.1 is finished.

In the following two subsections, we will combine Theorem 1.3 and Lemma 4.1 to get some interesting geometric results. The general idea is as follows. Theorem 1.3 gives us a condition on which Ω_x and $\Omega_{x'}$ are projectively equivalent, while Lemma 4.1 describes how we can write down defining equations of Ω_x in $\mathbb{P}M_n(\mathbb{C})$ when x satisfies the assumption in Lemma 4.1. Combining these two, one may get some nontrivial results in algebraic geometry. To illustrate how this machinery works, we discuss in the following subsections the case when the size of the matrices is $n = 2$ and $n = 3$. We give an easy geometric explanation for our results when $n = 2$. The results for $n = 3$ however are already nontrivial.

4.2. $n = 2$
We would like to write down the equation (26) explicitly. Let us distinguish three cases.

4.2.1. Case 1. In this case, we suppose $x \sim \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$. By Lemma 4.1, $\Omega_x = \{y \in \mathbb{P}M_2(\mathbb{C}): \exists x_0 \ s.t. \det(y - TI_2) = (x_0\lambda - T)^2\}$. Let $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$. Then for $y \in \Omega_x$ there must exist x_0 such that

$$\det \begin{pmatrix} y_1 - T & y_2 \\ y_3 & y_4 - T \end{pmatrix} = x = (x_0\lambda - T)^2. \tag{35}$$

Expanding the two sides of the equation as polynomials in T, we find

$$(y_1 y_4 - y_2 y_3) - (y_1 + y_4)T + T^2 = x_0^2 \lambda^2 - 2x_0\lambda T + T^2. \tag{36}$$

Comparing the coefficients, we get

$$\begin{cases} y_1 y_4 - y_2 y_3 = x_0^2 \lambda^2 \\ y_1 + y_4 = 2x_0\lambda \end{cases}. \tag{37}$$

Since x_0 is an undetermined variable, we eliminate it by combining the two above equations, and we find

$$y_1 y_4 - y_2 y_3 = \frac{1}{4}(y_1 + y_4)^2. \tag{38}$$

Hence, the defining equation of Ω_x for $x \sim \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$.

4.2.2. Case 2. In this case, we suppose that $x \sim \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, where $\lambda_1 \neq \lambda_2$. By Lemma 4.1, $\Omega_x = \{y \in \mathbb{P}M_2(\mathbb{C}): \exists x_{0 \ s.t.} \det(y - TI_2) = (x_0\lambda_1 - T)(x_0\lambda_2 - T)\}$. Again let $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$. Then for $y \in \Omega_x$ there must exist x_0 such that

$$\det \begin{pmatrix} y_1 - T & y_2 \\ y_3 & y_4 - T \end{pmatrix} = (x_0\lambda_1 - T)(x_0\lambda_2 - T). \tag{39}$$

Expanding both sides of the equation as polynomials in the variable T, we find

$$(y_1 y_4 - y_2 y_3) - (y_1 + y_4)T + T^2 = x_0^2 \lambda_1 \lambda_2 - x_0(\lambda_1 + \lambda_2)T + T^2, \tag{40}$$

which gives us

$$\begin{cases} y_1 y_4 - y_2 y_3 = x_0^2 \lambda_1 \lambda_2 \\ y_1 + y_4 = x_0(\lambda_1 + \lambda_2) \end{cases}, \tag{41}$$

by comparing the coefficients of the polynomials in T. Similarly, we eliminate the undetermined variable x_0 by combining these equations and find

$$(\lambda_1 + \lambda_2)^2(y_1 y_4 - y_2 y_3) = \lambda_1 \lambda_2(y_1 + y_4)^2. \tag{42}$$

Hence, the defining equation of Ω_x for $x \sim \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, where $\lambda_1 \neq \lambda_2$, is $(\lambda_1 + \lambda_2)^2(y_1 y_4 - y_2 y_3) = \lambda_1 \lambda_2(y_1 + y_4)^2$. Now we can apply Theorem 1.3 to get the following corollary.

Corollary 4.3. *Suppose that $\lambda_1 \neq \pm\lambda_2$ and $\lambda_1' \neq \pm\lambda_2'$. Then the two surfaces defined by $(\lambda_1 + \lambda_2)^2(y_1 y_4 - y_2 y_3) = \lambda_1 \lambda_2(y_1 + y_4)^2$ and $(\lambda_1' + \lambda_2')^2(y_1 y_4 - y_2 y_3) = \lambda_1'\lambda_2'(y_1 + y_4)^2$ are projectively equivalent in $\mathbb{P}^3$.*

Proof. Let $x \sim \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$ and $x' \sim \begin{pmatrix} \lambda_1' & \\ & \lambda_2' \end{pmatrix}$ with $\lambda_1 \neq \lambda_2$ and $\lambda_1' \neq \pm\lambda_2'$. By Theorem 1.3, we know that if there exist α, β satisfying $\alpha \neq 0$ and $\alpha \neq -3\beta$ such that

$$\begin{cases} \lambda_1' = \alpha\lambda_1 + \beta(\lambda_1 + \lambda_2) \\ \lambda_2' = \alpha\lambda_2 + \beta(\lambda_1 + \lambda_2) \end{cases}, \tag{43}$$

then Ω_x is projectively equivalent to $\Omega_{x'}$. By writing down the condition (43) explicitly, we find that $\lambda_1 \neq \pm\lambda_2$ and $\lambda_1' \neq \pm\lambda_2'$. Corollary 4.3 follows, as the defining equations for Ω_x and $\Omega_{x'}$ were calculated above.

Remark. In fact, Corollary 4.3 can be proved without the machinery that we developed in the article. It can be viewed as a direct consequence of the following well-known algebraic geometry fact [6]:

Let Y and Y' be *smooth* quadric hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. Then Y and Y' are projectively equivalent in $\mathbb{P}^n(\mathbb{C})$.

One checks readily that under the assumption of Corollary 4.3, the quadric surfaces in question are smooth. Thus they are projectively equivalent in $\mathbb{P}^3(\mathbb{C})$.

4.2.3. Case 3. In this case, we consider $x \sim \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$. This case is trivial: $\Omega_x = G_x = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset \mathbb{P}^3(\mathbb{C})$.

4.3. $n = 3$
We need to write down the equation (26) explicitly when the size of the matrices is $n = 3$.

4.3.1. Case 1. In this case, we suppose that $x \sim \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$. By Lemma 4.1, $\Omega_x = \{y \in \mathbb{P}M_3(\mathbb{C}): \exists x_0 \ s.t. \det(y - TI_2) = (x_0\lambda - T)^3\}$. Let $y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$. Then for $y \in \Omega_x$ there must exist x_0 such that

$$\det \begin{pmatrix} y_{11} - T & y_{12} & y_{13} \\ y_{21} & y_{22} - T & y_{23} \\ y_{31} & y_{32} & y_{33} - T \end{pmatrix} = (x_0\lambda - T)^3. \tag{44}$$

Expanding both sides of the equation as polynomials in T, we find

$$(y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11})$$
$$+ (y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{22} - y_{22}y_{33})T + (y_{11} + y_{22} + y_{33})T^2 - T^3 \tag{45}$$

$$= x_0^3\lambda^3 - 3x_0^2\lambda^2 T + 3x_0\lambda T^2 - T^3.$$

Comparing the coefficients, we have

$$\begin{cases} y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11} = x_0^3\lambda^3 \\ y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{22} - y_{22}y_{33} = -3x_0^2\lambda^2 \\ y_{11} + y_{22} + y_{33} = 3x_0\lambda. \end{cases} \tag{46}$$

Eliminating the undetermined variable x_0, we get

$$\begin{cases} y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11} = \frac{1}{27}(y_{11} + y_{22} + y_{33})^3 \\ y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{22} - y_{22}y_{33} = -\frac{1}{3}(y_{11} + y_{22} + y_{33})^2. \end{cases} \tag{47}$$

The above system of equations defines Ω_x in $\mathbb{P}^8(\mathbb{C})$ for $x \sim \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$.

4.3.2. Case 2. In this case, we suppose that $x \sim \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$ with $\lambda_1 \neq \lambda_2$. By Lemma 4.1, $\Omega_x = \{y \in \mathbb{P}M_3(\mathbb{C}): \exists x_0 \ s.t. \det(y - TI_3) = (x_0\lambda_1 - T)^2(x_0\lambda_2 - T)\}$. Let $y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$. Then for $y \in \Omega_x$ there must exist x_0 such that

$$\det \begin{pmatrix} y_{11} - T & y_{12} & y_{13} \\ y_{21} & y_{22} - T & y_{23} \\ y_{31} & y_{32} & y_{33} - T \end{pmatrix} = (x_0\lambda_1 - T)^2(x_0\lambda_2 - T). \tag{48}$$

Expanding both sides as polynomials in T, we find

$$(y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11})$$
$$+ (y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{22} - y_{22}y_{33})T + (y_{11} + y_{22} + y_{33})T^2 - T^3 \tag{49}$$

$$= x_0^3 \lambda_1^2 \lambda_2 - x_0^2(2\lambda_1\lambda_2 + \lambda_1^2)T + x_0(2\lambda_1 + \lambda_2)T^2 - T^3.$$

Comparing the coefficients of both sides, we get the following system of equations.

$$\begin{cases} y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11} = x_0^3\lambda^3 \\ y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{22} - y_{22}y_{33} = -x_0^2(2\lambda_1\lambda_2 + \lambda_1^2) \\ y_{11} + y_{22} + y_{33} = x_0(2\lambda_1 + \lambda_2). \end{cases} \quad (50)$$

we eliminate the undetermined variable $_{x0}$ and get the following system of equations.

$$\begin{cases} (2\lambda_1 + \lambda_2)^3(y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11}) \\ \qquad = \lambda_1^2\lambda_2(y_{11} + y_{22} + y_{33})^3 \\ (2\lambda_1 + \lambda_2)^2(y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{33} - y_{22}y_{33}) \\ \qquad = -(2\lambda_1\lambda_2 + \lambda_1^2)(y_{11} + y_{22} + y_{33})^2. \end{cases} \quad (51)$$

This is the system of equations that defines $\Omega_x \subset \mathbb{P}^8(\mathbb{C})$ for $x \sim \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$. Following the

notation given in the Introduction, we let Y_{λ_1,λ_2} denote the projective variety defined by this system of equations. We can now use Theorem 1.3 to prove the following.

Corollary 4.4. *Given two pairs of distinct complex numbers (λ_1, λ_2) and (λ_1', λ_2'), suppose that*
* *either (λ_1, λ_2) is proportional to (λ_1', λ_2') up to a nonzero constant,*
* *or the following conditions hold*

$$\begin{cases} \lambda_2 \neq -2\lambda_1 \\ (2\lambda_1 + \lambda_2)\lambda_1' \neq \lambda_2\lambda_2' \\ 2(2\lambda_1 + \lambda_2)\lambda_1' \neq (3\lambda_1 - \lambda_2)\lambda_2' \end{cases} \quad (52)$$

Then the projective varieties Y_{λ_1,λ_2} and $Y_{\lambda_1',\lambda_2'}$ are projectively equivalent in $\mathbb{P}^8$.

Proof. Let $x \sim \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$ and $x' \sim \begin{pmatrix} \lambda_1' & 1 & \\ & \lambda_1' & \\ & & \lambda_2' \end{pmatrix}$, with $\lambda_1 \neq \lambda_2$ and $\lambda_1' \neq \lambda_2'$. By Theorem 1.3,

we know that if there exist α, β that satisfy $\alpha \neq 0$ and $\alpha \neq -2\beta$ such that

$$\begin{cases} \lambda_1' = \alpha\lambda_1 + \beta(2\lambda_1 + \lambda_2) \\ \lambda_2' = \alpha\lambda_2 + \beta(2\lambda_1 + \lambda_2)' \end{cases} \quad (53)$$

then Ω_x is projectively equivalent to $\Omega_{x'}$. By writing down the condition (53) explicitly, we find that the three listed conditions allow for the existence of such α and β. Corollary 4.4 follows as the defining equations for Ω_x and $\Omega_{x'}$ were calculated above.

4.3.3. Case 3. In this case, we suppose that $x \sim \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$ with $\lambda_1, \lambda_2, \lambda_3$ distinct. By Lemma 4.1,

$\Omega_x = \{y \in \mathbb{P}M_3(\mathbb{C}): \exists x_0 \ s.t. \det(y - TI_3) = (x_0\lambda_1 - T)(x_0\lambda_2 - T)(x_0\lambda_3 - T)\}$. Let $y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$. Then for $y \in \Omega_x$ there must exist x_0 such that

$$\det \begin{pmatrix} y_{11} - T & y_{12} & y_{13} \\ y_{21} & y_{22} - T & y_{23} \\ y_{31} & y_{32} & y_{33} - T \end{pmatrix} = (x_0\lambda_1 - T)(x_0\lambda_2 - T)(x_0\lambda_3 - T). \quad (54)$$

Expanding both sides as polynomials in T, we find

$$(y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11})$$
$$+ (y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{22} - y_{22}y_{33})T + (y_{11} + y_{22} + y_{33})T^2 - T^3$$
$$(55)$$

$$= x_0^3\lambda_1\lambda_2\lambda_3 - x_0^2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)T + x_0(\lambda_1 + \lambda_2 + \lambda_3)T^2 - T^3.$$

Comparing the coefficients of both sides, we get the following system of equations.

$$\begin{cases} y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11} = x_0^3\lambda_1\lambda_2\lambda_3 \\ y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{22} - y_{22}y_{33} = -x_0^2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) \\ y_{11} + y_{22} + y_{33} = x_0(\lambda_1 + \lambda_2 + \lambda_3). \end{cases}$$

we eliminate the undetermined variable x_0 and get the following system of equations.

$$\begin{cases} (\lambda_1 + \lambda_2 + \lambda_3)^3(y_{11}y_{22}y_{33} + y_{12}y_{23}y_{31} + y_{21}y_{32}y_{13} - y_{31}y_{13}y_{22} - y_{21}y_{12}y_{33} - y_{32}y_{23}y_{11}) \\ \qquad = \lambda_1\lambda_2\lambda_3(y_{11} + y_{22} + y_{33})^3 \\ (\lambda_1 + \lambda_2 + \lambda_3)^2(y_{31}y_{13} + y_{21}y_{12} + y_{32}y_{23} - y_{11}y_{22} - y_{11}y_{33} - y_{22}y_{33}) \\ \qquad = -(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)(y_{11} + y_{22} + y_{33})^2. \end{cases} \qquad (56)$$

The above system of equations defines Ω_x in $\mathbb{P}^8(\mathbb{C})$ for $x \sim \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$ with $\lambda_1, \lambda_2, \lambda_3$ distinct.

Following similar notation as given in the previous case, we define a projective variety $Y_{\lambda_1,\lambda_2,\lambda_3}'$ in $\mathbb{P}^8$ of points with homogeneous coordinates $[y_{11}: y_{12}: y_{13}: y_{21}: y_{22}: y_{23}: y_{31}: y_{32}: y_{33}]$ satisfying the system of equations in (56).

Corollary 4.5. *Given two triples of distinct complex numbers* $(\lambda_1, \lambda_2, \lambda_3)$ *and* $(\lambda_1', \lambda_2', \lambda_3')$, *suppose that*

• *either* $(\lambda_1, \lambda_2, \lambda_3)$ *is proportional to* $(\lambda_1', \lambda_2', \lambda_3')$ *up to a nonzero constant,*

• *or the following conditions hold*

$$\begin{cases} \lambda_1'\lambda_2 - \lambda_1\lambda_2' + \lambda_2'\lambda_3 - \lambda_2\lambda_3' + \lambda_3'\lambda_1 - \lambda_3\lambda_1' = 0 \\ \lambda_1 + \lambda_2 + \lambda_3 \neq 0 \\ \lambda_1' + \lambda_2' + \lambda_3' \neq 0 \\ 2\lambda_1 \neq \lambda_2 + \lambda_3 \\ 2\lambda_1' \neq \lambda_2' + \lambda_3' \end{cases}$$

Then the projective varieties $Y_{\lambda_1,\lambda_2,\lambda_3}'$ *and are projectively equivalent in* $\mathbb{P}^8$.

Proof. Let $x \sim \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$ and $x' \sim \begin{pmatrix} \lambda_1' & & \\ & \lambda_2' & \\ & & \lambda_3' \end{pmatrix}$, with $\lambda_1, \lambda_2, \lambda_3$ distinct (resp. $\lambda_1', \lambda_2', \lambda_3'$). By Theorem 1.3, we know that if there exist α, β that satisfy $\alpha \neq 0$ and $\alpha \neq -3\beta$ such that

$$\begin{cases} \lambda_1' = \alpha\lambda_1 + \beta(\lambda_1 + \lambda_2 + \lambda_3) \\ \lambda_2' = \alpha\lambda_2 + \beta(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_3' = \alpha\lambda_3 + \beta(\lambda_1 + \lambda_2 + \lambda_3) \end{cases} \qquad (57)$$

then Ω_x is projectively equivalent to $\Omega_{x'}$. Given the five listed conditions, we find that there exist $\alpha = \frac{2\lambda_1' - \lambda_2' - \lambda_3'}{2\lambda_1 - \lambda_2 - \lambda_3}$ and satisfying condition (57). Corollary 4.5 follows as the defining equations for Ω_x and $\Omega_{x'}$ were calculated above.

5. Conclusion

Although the classification of orbits for any algebraic group action on projective spaces seems to be out of reach for the moment, we manage to give efficient criteria to determine when two orbits are projectively equivalent for the four families of explicit examples we presented in the Introduction. We also find interesting geometric applications of our research.

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