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# Explicit form of Laplace-Beltrami operator on SO(3) in the view of Fourier analysis

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**Abstract.** Fourier analysis plays a central role in the modern physics, engineering, and mathematics itself. In the field of differential geometry, a Lie group G gives a symmetric structure, and one may apply the Fourier analysis by means of matrix-valued irreducible representations. Even though the entries of these irreducible representations are already shown to be the eigenfunctions of the Laplace-Beltrami operator, it is still desirable to consider a concrete example where both the operator and the irreducible representations can be computed explicitly. This study gives an explicit form of the Laplace-Beltrami operator on SO(3) using direct computations and show also that each entry of the irreducible representations $o_n^{ij}$ is indeed an eigenfunction of this operator. Therefore, one can also find the application of the Fourier Analysis on differential equations, in this study Poisson's equation as an example, using the Laplace-Beltrami operator as the corresponding differential operator. Overall, these results shed light on guiding further exploration of Fourier analysis.

**Keywords:** Fourier analysis on SO(3), Laplace-Beltrami operator on SO(3), poisson's equation.

## 1. Introduction

Fourier analysis is frequently used in various mathematical aspects. In the case of the Euclidean space $\mathbb{R}^n$, one can define the Fourier transform as in the Chapter 5 of Stein's Fourier analysis [1]. The extreme importance of Fourier analysis on solving differential equations is that one may find an orthonormal basis $\{b_j(x)\}$ of $L^2(\mathbb{R}^n)$, and in more general cases $L^2(M)$ for $M$ being a measure space, such that this basis is also a family of eigenfunctions for some operators $T: L^2(M) \to L^2(M)$ (especially differential operators). Then if there is an equation $T(u) = f$, one may take the Fourier expansion, and use the fact that $b_i$ is an eigenfunction of $T$ to obtain the new equation that is easier to solve:

$$\hat{u}(j)a_j b_j(x) = \hat{f}(j)b_j(x) \tag{1}$$

Finding such basis can be done quite directly in $L^2(\mathbb{R}^n)$, one basis with $T = \partial_{x_1}^2 + \cdots + \partial_{x^n}^2$ is $b_j(x) = e^{2\pi i(j\cdot x)}$, where $j \in \mathbb{R}^n$. One may be ambitious to consider the more general case when $M$ is a Lie group $GL(n)$ (such that $M$ is a smooth manifold where one can define functions on it and tangent vectors as differential operators on it).This is no mean feat. Because in general the matrix multiplication is not commutative, one can no longer use $b_j(x)$ that is an element on the complex unit circle, for this would be commutative.

The difficulty does not only appear in the choice of orthonormal basis, but also appear in the choice of differential operator, and in this passage the Laplace-Beltrami operator. Alexander Grigor'yan contributed a lot to the analysis on Riemannian manifolds, and in his papers, he considered many abstract measure spaces and differential operators [2, 3]. He analyzed their consequent eigenvalues, properties of measures, capacities, and volume growth.

In Sugira's work, many useful properties were introduced when the Fourier analysis on Lie groups was considered, and by Shur's lemma, he also showed that each entry of the irreducible representations of a Lie group is an eigenfunction of the corresponding Laplace-Beltrami operator defined in the usual way [4]. However, this study will not continue these evaluations of abstract properties, and instead consider a concrete example of a measure space $SO(3)$, and the explicit expression of the Laplace-Beltrami operator defined on this space. According to Dym and Vilenkin, the Fourier analysis in special cases when $M = SO(3)$, and $M = SU(2)$ are considered [5, 6]. Unfortunately, Dym didn't give the definition of a Laplace-Beltrami operator on $SO(3)$ and therefore missed a fruitful relationship between Fourier analysis and differential equations, which this study should now consider.

In section 2, preliminaries about the Fourier analysis on $SO(3)$ are listed. There is a parametrization of $SO(3)$ will be found, and therefore the integration will be defined. Useful properties that the integration is left- and right-translation invariant are proved there. Then this study will arrive at the Fourier analysis, which is considering $(2n+1)\times(2n+1)$ matrices $o_n$ instead of the characters used in commutative cases, such as $e^{2\pi i(j\cdot x)}$ in $\mathbb{R}^n$. A property that the Fourier transform actually converts the algebra of convolutions to the algebra of multiplications is derived at the end of section 2, which is conducive to the solutions of differential equations in section 4. The desired Laplace-Beltrami operator is defined in section 3.1, with quite explicit formulas and computations. Section 3.2 also contains the fact that each entry of $o_n$ is an eigenfunction of the Laplace-Beltrami operator defined in section 3.1. Section 4 will utilize this fact of $o_n$, and derive solutions to some important differential equations on $SO(3)$. Therefore, since the sphere $S^2$ is simply a subspace of $SO(3)$, one may derive the solutions to the corresponding differential equations on $S^2$.

## 2. Preparations

### *2.1. The parametrization of SO(3)*

The parametrization of a matrix in SO(3) used in the passage is given by the Euler angle $(\varphi,\theta,\psi)$, where an arbitrary matrix h can be written as

$$h(\varphi,\theta,\psi)=\begin{bmatrix}\cos\theta\cos\psi\cos\varphi-\sin\psi\sin\varphi & -\cos\theta\sin\psi\cos\varphi-\cos\psi\sin\varphi & \sin\theta\cos\varphi\\ \cos\theta\cos\psi\sin\varphi+\sin\psi\cos\varphi & \cos\psi\cos\varphi-\cos\theta\sin\psi\sin\varphi & \sin\theta\sin\varphi\\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta\end{bmatrix} \quad (2)$$

with $h(\cdot,\cdot,\cdot)\colon[0,2\pi]\times[0,\pi]\times[0,2\pi]$ the matrix-valued identification map. Therefore, if one knows each entry $h_{ij}$ of a matrix, its Euler angle is given by

$$(\varphi,\theta,\psi)=(\arctan\frac{h_{23}}{h_{13}},\arccos h_{33},\arctan(-\frac{h_{32}}{h_{31}})) \quad (3)$$

### *2.2. Integration on SO(3)*

A measure $\mu$ on $SO(3)$ is a map from measurable sets to non-negative real numbers, such that $\mu(\bigcup_{n=0}^{\infty}E_n)=\sum_{n=0}^{\infty}\mu(E_n)$, where $E_i$ are disjoint measurable sets [7]. Using the parametrization given in section 2.1, one may define such measure by means of product measure $\mu(A\times B)=\mu_1(A)\mu_2(B)$, for $A\in\{h\in SO(3)|h=h(\varphi,\theta,0)\}$, and $B\in\{h\in SO(3)|h=h(0,0,\psi)\}$. Therefore, the integration is defined to be $\int_{SO(3)}f(g)dg=\frac{1}{4\pi}\int_{S^2}\left(\frac{1}{2\pi}\int_0^{2\pi}f\big(gh(0,0,\psi)\big)d\psi\right)d\sigma(x)$. The constant $\frac{1}{4\pi}$ and $\frac{1}{2\pi}$ is made so that $\int_{SO(3)}dg=1$. An important observation is that the integration defined in this way is both right- and left-translation invariant, which is

$$\int_{SO(3)}f(gh)dg=\int_{SO(3)}f(hg)dg=\int_{SO(3)}f(g)dg \quad (4)$$

The proof is given by Dym, and this study will use the above property in the following sections [5].

### *2.3. Fourier analysis on SO(3)/K*

Following the steps of Dym [5], K is defined to be $\{h \in SO(3) | h = h(0,0,\psi)\}$. $SO(3)/K$ the space of cosets. An orthonormal basis of $L^2\ (SO(3)/K)$ of extremely importance is $e_n^l(x)$, which is constructed as follows. Let $p_n(x)$ be the Legendre polynomial, and define $\tilde{p}_n(h) = p_n(\cos\theta)$, where $\theta$ is given by $h(\varphi,\theta,\psi)$. $\tilde{p}_n^g(h) = \tilde{p}_n(gh)$ for all $g \in SO(3)$ span a subspace $M_n$ of $L^2\ (SO(3)/K)$. Picking orthonormal basis $e_n^l(x)$ of $M_n$ gives the Fourier expansion of $\tilde{p}_n^g(x)$ by $\sum_l \int_G \tilde{p}_n^g(h)\overline{e_n^l}(h)dh\, e_n^l(x)$. The computation that Dym did gives us the addition formula [5]

$$\tilde{p}_n(g^{-1}h) = \frac{1}{2n+1}\sum_l \overline{e_n^l}(g)e_n^l(h) \tag{5}$$

By the above formula, the dimension of $M_n$ is 2n+1, and the following expression can be established.

$$e_n^l(g) = (2n+1)\int_{SO(3)} \tilde{p}_n(g^{-1}h)e_n^l(h)dh = (2n+1)\int_{SO(3)} \tilde{p}_n(h^{-1}g)e_n^l(h)dh \tag{6}$$

The last equality is because for a function $f: SO(3) \to \mathbb{R}$ that is constant on each double coset $KgK$, $f(g) = f(g^{-1})$, for arbitrary $g \in SO(3)$. This is indeed the case for $\tilde{p}_n$ since the value of $\tilde{p}_n$ is only determined by $\theta$.

### *2.4. Fourier analysis on SO(3)*

For $g \in SO(3)$, a matrix $o_n(g)$ is defined in Dym's Fourier Analysis such that

$$o_n^{ij}(g) = \int_{SO(3)} e_n^i(gh)\overline{e_n^j}(h)\, dh \tag{7}$$

matrices defined in this way will be considered irreducible representations, and therefore a function $f \in L^2(SO(3))$ has the following expansion

$$f(g) = \sum_{n=0}^{\infty}(2n+1)tr\left(\hat{f}(n)o_n(g)\right) \tag{8}$$

with $\hat{f}$ be a $(2n+1) \times (2n+1)$ matrix, where each entry is defined to be

$$\hat{f}(n)^{ij} = \int_{SO(3)} f(g)\overline{o_n^{ji}}(g)dg \tag{9}$$

or equivalently

$$\hat{f}(n) = \int_{SO(3)} f(g)o_n^{\#}(g)dg \tag{10}$$

where $(\cdot)^{\#}$ is denoted as the conjugate transpose of a matrix. The proof that every function in $L^2(SO(3))$ can be written in this expansion is given by Dym in his Fourier Analysis [5]. Here, the proof of $\widehat{(f_1 * f_2)} = \widehat{f_2}\widehat{f_1}$ is given in addition, where $*$ is the convolution defined as $f * g = \int_{SO(3)} f(kh^{-1})g(h)dh$.One notices that $\sum_{k=-n}^{n} o_n^{ik}(g)o_n^{kj}(h) = o_n^{ij}(gh)$, where the first equality is given by formula (7), the fourth equality is simply the usage of addition formula, and the fifth equality utilizes the fact that $\tilde{p}_n$ is constant on each double coset, so $\tilde{p}_n(g) = \tilde{p}_n(g^{-1})$. The sixth equality is given by formula (6) and $\tilde{p}_n$ being real-valued function, and using the translation invariant property of the integral, one finds the seventh equality. Now, with the multiplicative property given above, one can show $\widehat{(f_1 * f_2)} = \widehat{f_2}\widehat{f_1}$.

## 3. Explicit form of Laplace-Beltrami operator on SO(3)

It is well-known that a Laplace-Beltrami operator on an arbitrary manifold can be defined by the following equation

$$\Delta := \frac{1}{\sqrt{\det g}}\partial_{\mathrm{i}}\sqrt{\det g}\ g^{\mathrm{ij}}\,\partial_{\mathrm{j}}\,, \tag{11}$$

where $g$ is a matrix valued function corresponds to a Riemannian metric $(\cdot\,,\ \cdot)_g$ such that $(v,w)_{\mathrm{g}} = (\sum_{\mathrm{i,j}\le 3}\ g_{ij}dx^i dx^j)vw$ [8], the determinant of $g$ is denoted by $\det g$, and each entry of its inverse matrix is denoted by $g^{ij}$.

### *3.1. An explicit formula of the Laplace-Beltrami operator on SO(3)*

Using the exponential map $e^{(\cdot)}: SO(3) \to SO(3)$ ,where $e^h = \sum_{n=0}^{\infty}\frac{h^n}{n!}$, together with the Lie algebra $\mathfrak{g} \in \mathfrak{so}(3)$ on SO(3), the vector space $T_hSO(3)$ at $h$ is isomorphic to $h\mathfrak{so}(3) = \{h\mathfrak{g}|\mathfrak{g} \in \mathfrak{so}(3)\}$[9], so an arbitrary vector field $L_{\mathfrak{g}}: C^\infty(SO(3)) \to C^\infty(SO(3))$ is defined by $L_{\mathfrak{g}}f(h) := \frac{d}{dt}f(he^{t\mathfrak{g}})|_{t=0}$, for a smooth function $f \in C^\infty(SO(3))$ and matrix $h \in SO(3)$ [10]. The basis of $\mathfrak{so}(3)$ is as follows:

$$\mathfrak{g}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathfrak{g}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \mathfrak{g}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{12}$$

It is not hard to compute

$$e^{\mathfrak{g}_1 t} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, e^{\mathfrak{g}_2 t} = \begin{bmatrix} \cos \mathrm{t} & 0 & \sin \mathrm{t} \\ 0 & 1 & 0 \\ -\sin \mathrm{t} & 0 & \cos \mathrm{t} \end{bmatrix}, e^{\mathfrak{g}_3 t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \mathrm{t} & -\sin \mathrm{t} \\ 0 & \sin \mathrm{t} & \cos \mathrm{t} \end{bmatrix} \tag{13}$$

By denoting the tangent vector $\partial_i(f(h)) = L_{\mathfrak{g}_\mathrm{i}}f(h) = \frac{d}{dt}f(he^{\mathfrak{g}_\mathrm{i}\mathrm{t}})|_{t=0}$ at an arbitrary point $h \in SO(3)$, $(\partial_1, \partial_2, \partial_3)$ is indeed a global frame for the space of vector fields on $SO(3)$. That is, each vector field $L_{\mathfrak{g}}: C^\infty(SO(3)) \to C^\infty(SO(3))$ can be written as a linear combination of $L_{\mathfrak{g}_\mathrm{i}}$. This is indeed the case, because by considering $\mathfrak{g} = \mathfrak{g}_1 t_1 + \mathfrak{g}_2 t_2 + \mathfrak{g}_3 t_3$ , $L_{\mathfrak{g}}f(h) = \frac{d}{dt}f(he^{\mathfrak{g}\mathrm{t}})|_{t=0} = (t_1 L_{\mathfrak{g}_1} + t_2 L_{\mathfrak{g}_2} + t_3 L_{\mathfrak{g}_3})(f)(h)$ . The Riemannian metric used in this study is given by $(v,w)_g = tr(j(v)^t j(w))$. Hence, $g_{ij}$ is given by $tr(\mathfrak{g}_\mathrm{i}^t \mathfrak{g}_\mathrm{j}) = 2\delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol. Therefore, the Laplace-Beltrami operator can be simplified as

$$\Delta = \frac{1}{\sqrt{\det g}}\partial_i \sqrt{\det g}\, g^{ij}\partial_j = \frac{1}{2}(\partial_1^2 + \partial_2^2 + \partial_3^2) \tag{14}$$

It is desirable to compute $\partial_1, \partial_2, \partial_3$respectively. Using the parametrization defined in section 2, the computation is as follows. Since $\widetilde{h_\iota} = he^{\mathfrak{g}_\mathrm{i}\mathrm{t}}$ is a matrix in SO(3), one can find its Euler angle $(\widetilde{\varphi_\iota}, \widetilde{\theta_\iota}, \widetilde{\psi_\iota})$ defined in section 2 given by $\left(\arctan\frac{\widetilde{h_{\iota 23}}}{\widetilde{h_{\iota 13}}}, \arccos \widetilde{h_{\iota 33}}, \arctan\left(-\frac{\widetilde{h_{\iota 32}}}{\widetilde{h_{\iota 31}}}\right)\right)$. Therefore,

$$\partial_i f = \frac{d}{dt}f(he^{\mathfrak{g}_\mathrm{i}\mathrm{t}})|_{t=0} = \frac{\partial}{\partial\varphi}f\frac{d\widetilde{\varphi_\iota}}{dt}|_{t=0} + \frac{\partial}{\partial\theta}f\frac{d\widetilde{\theta_\iota}}{dt}|_{t=0} + \frac{\partial}{\partial\psi}f\frac{d\widetilde{\psi_\iota}}{dt}|_{t=0} \tag{15}$$

For $\mathfrak{g}_1$, this is simple, since

$$\widetilde{h_1} = (\varphi, \theta, \psi + t). \tag{16}$$

and the corresponding Euler angle is given by $(\widetilde{\varphi_1}, \widetilde{\theta_1}, \widetilde{\psi_1}) = (\varphi, \theta, \psi + t)$. Therefore, $\partial_1$is computed directly as follows.

$$\partial_1 = \frac{d\widetilde{\varphi_1}}{dt}|_{t=0}\frac{\partial}{\partial\varphi} + \frac{d\widetilde{\theta_1}}{dt}|_{t=0}\frac{\partial}{\partial\theta} + \frac{d\widetilde{\psi_1}}{dt}|_{t=0}\frac{\partial}{\partial\psi} = \frac{\partial}{\partial\psi} \tag{17}$$

For $\mathfrak{g}_2$, the computation would be harder.

$$\widetilde{h_2} = \begin{bmatrix} \widetilde{h_{2_{11}}} & \widetilde{h_{2_{12}}} & \widetilde{h_{2_{13}}} \\ \widetilde{h_{2_{21}}} & \widetilde{h_{2_{22}}} & \widetilde{h_{2_{23}}} \\ \widetilde{h_{2_{31}}} & \widetilde{h_{2_{32}}} & \widetilde{h_{2_{33}}} \end{bmatrix} \tag{18}$$

where $\widetilde{h_{2_{11}}}$ , $\widetilde{h_{2_{12}}}, \widetilde{h_{2_{21}}}, \widetilde{h_{2_{22}}}$ are of no interest, and the other entries are given by $\widetilde{h_{2_{13}}} = \sin t\,(\cos\theta\cos\psi\cos\varphi - \sin\psi\sin\varphi) + \sin\theta\cos \mathrm{t}\cos\varphi$ , $\widetilde{h_{2_{23}}} = \sin t\,(\cos\theta\cos\psi\sin\varphi + \sin\psi\cos\varphi) + \sin\theta\cos \mathrm{t}\sin\varphi$ , $\widetilde{h_{2_{31}}} = -\sin\theta\cos \mathrm{t}\cos\ \psi - \cos\theta\sin \mathrm{t}$, $\widetilde{h_{2_{32}}} = \sin\theta\sin\psi$ , $\widetilde{h_{2_{33}}} =$

$\cos\theta\cos t - \sin\theta\sin t\cos\psi$ and therefore $\frac{d\widetilde{\varphi_2}}{dt}|_{t=0} = \csc\theta\sin\psi$, $\frac{d\widetilde{\theta_2}}{dt}|_{t=0} = \cos\psi$, $\frac{d\widetilde{\psi_2}}{dt}|_{t=0} = -\cot\theta\sin\psi$. Using these data, Eq. (15) shows that

$$\partial_2 = \csc\theta\sin\psi\frac{\partial}{\partial\varphi} + \cos\psi\frac{\partial}{\partial\theta} - \cot\theta\sin\psi\frac{\partial}{\partial\psi} \quad (19)$$

Similarly for $\mathfrak{g}_3$, computing corresponding $\widetilde{h_3}_{ij}$ finds that

$$\partial_3 = -\csc\theta\cos\psi\frac{\partial}{\partial\varphi} + \sin\psi\frac{\partial}{\partial\theta} + \cot\theta\cos\psi\frac{\partial}{\partial\psi} \quad (20)$$

Due to the computations above, the Laplace-Beltrami operator will finally be computed explicitly.

$$\Delta = \frac{1}{2}(\partial_1^2 + \partial_2^2 + \partial_3^2) = \frac{1}{2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\psi^2} - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \quad (21)$$

### *3.2. Egenfunctions of the Laplace-Beltrami operator on SO(3)*

Before checking each entry of $o_n$ defined in section 2 is indeed the eigenfunction of the Laplace-Beltrami operator defined in this way, one should first note that $\Delta$ commutes with the right shift operator $R_g: C^\infty(SO(3)) \to C^\infty(SO(3))$, where $R_g f(h) = f(hg)$, and left shift operator $L_g f(h) = f(gh)$, this is the essential reason why this study introduce Laplace-Beltrami operator in this way. The proof is as follows. By revisiting the definition of $\partial_i$, which is $\partial_i(f(h)) = \frac{d}{dt}f(he^{\mathfrak{g}_i t})|_{t=0}$, one has by considering $\tilde{f}(t) = f(he^{\mathfrak{g}_i t})$ and using Taylor expansion,

$$f(he^{\mathfrak{g}_i t}) = \tilde{f}(t) = \sum_{n=0}^{\infty}\frac{1}{n!}t^n\frac{d^n}{dt^n}\tilde{f}(0) = \left(\sum_{n=0}^{\infty}\frac{1}{n!}t^n\frac{d^n}{dt^n}\right)f(h) = \left(\sum_{n=0}^{\infty}\frac{1}{n!}t^n\partial_i^n\right)f(h) \quad (22)$$

By finding a parametrization $g = e^{\mathfrak{g}_1\varphi}e^{\mathfrak{g}_2\theta}e^{\mathfrak{g}_1\psi}$, one can therefore write the right shift operator as

$$R_g f(h) = f(hg) = f\left(he^{\mathfrak{g}_1\varphi}e^{\mathfrak{g}_2\theta}e^{\mathfrak{g}_1\psi}\right) = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\psi^n\partial_1^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\theta^n\partial_2^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\varphi^n\partial_1^n\right)f(h) \quad (23)$$

and if one denotes $Iv(f)(h) = f^-(h) = f(h^{-1})$ the left shift operator as

$$L_g f(h) = \left(\left(\sum_{n=0}^{\infty}\frac{1}{n!}-\varphi^n\partial_1^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}-\theta^n\partial_2^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}-\psi^n\partial_1^n\right)Iv(f)\right)(h^{-1}) \quad (24)$$

Here, Lie bracket is defined to be $[X,Y] = XY - YX$, where X and Y are tangent vectors of a lie group. This means that $XY = YX + [X,Y]$. It is easy to see the properties $[X+Z,Y] = [X,Y] + [Z,Y]$, and $[X,Y] = -[Y,X]$. According to this definition, one can compute directly that

$$[\partial_1,\partial_2] = -\partial_3, [\partial_2,\partial_3] = -\partial_1, [\partial_3,\partial_1] = -\partial_2 \quad (25)$$

Granted these data, one has

$$[\Delta,\partial_1] = \frac{1}{2}([\partial_1^2,\partial_1] + [\partial_2^2,\partial_1] + [\partial_3^2,\partial_1]) = 0 \quad (26)$$

Similarly, one has $[\Delta,\partial_2] = 0, [\Delta,\partial_3] = 0$. Now, one can prove the commutativity between Laplace-Beltrami operator and right-, left-shift operators

$$\Delta R_g f(h) = R_g\Delta f(h) \quad (27)$$

and since

$$\Delta Iv(f(h)) = \Delta f\big(h(-\psi,-\theta,-\varphi)\big) = Iv\big(\Delta f(h)\big) \quad (28)$$

one therefore has

$$\Delta L_g f(h) = \left(\left(\sum_{n=0}^{\infty}\frac{1}{n!}-\varphi^n\partial_1^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}-\theta^n\partial_2^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}-\psi^n\partial_1^n\right)Iv(\Delta f)\right)(h^{-1}) = L_g\Delta f(h) \quad (29)$$

An observation of interest is that each entry of a matrix $o_n$ defined in section 2 as a representation of $SO(3)$ is indeed an eigenfunction of the Laplace-Beltrami operator defined in this section. To start with,

$$\Delta\tilde{p}_n(g) = \frac{1}{2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}p_n(\cos\theta)\right) = \frac{1}{2}\big(-n(n+1)p_n(\cos\theta)\big) \quad (30)$$

The last equality is due to the recurrence relation of Legendre polynomials. Now one can compute the Laplacian of $o_n^{ij}(g)$ directly as follows.

$$\Delta o_n^{ij}(g) = \frac{(-n(n+1))}{2}o_n^{ij}(g) \quad (31)$$

The first and third equality is the formula (7) and (6) respectively. Then the commutativity of Laplace-Beltrami operator and left-/right- shift operator is used to get the fifth equality. Finally, the Eq. (30) is used at the sixth equality.

## 4. Solutions to the differential equations using Laplace-Beltrami operator & Fourier analysis

### *4.1. Solutions to the Poisson's equation*

Poisson's equation is defined to be $\Delta u(g) = f(g)$, since 3-dimensional manifold $SO(3)$ has no boundary. Then, according to Eqs. (8) and (9), one has

$$\Delta \sum_{n=0}^{\infty}(2n+1)tr\big(\hat{u}(n)o_n(g)\big) = \sum_{n=0}^{\infty}(2n+1)tr\left(\hat{f}(n)o_n(g)\right) \tag{32}$$

However, since the $tr(ho_n(g))$, which represents the trace of the matrix $ho_n(g)$, is a linear combination of entries $o_n^{ij}(g)$ for all $(2n+1)\times(2n+1)$ matrices

$$\Delta \sum_{n=0}^{\infty}(2n+1)tr\big(\hat{u}(n)o_n(g)\big) = \sum_{n=0}^{\infty}(2n+1)tr\left(\frac{(-n(n+1))}{2}\hat{u}(n)o_n(g)\right) \tag{33}$$

Because the Fourier expansion is unique, one gets

$$\hat{u}(n) = \frac{2}{-n(n+1)}\hat{f}(n) \tag{34}$$

then

$$\Delta u = \sum_{n=0}^{\infty}(2n+1)tr\left(\hat{f}(n)o_n(g)\right) = f \tag{35}$$

So, a solution of $\Delta u = f$ is given by

$$u = \sum_{n=0}^{\infty}(2n+1)tr\left(\frac{2}{-n(n+1)}\int_{SO(3)} f(g)o_n^{\#}(g)dg\, o_n(g)\right) \tag{36}$$

One may further define

$$\Phi(g) = \sum_{n=0}^{\infty}(2n+1)\frac{2}{-n(n+1)}tr\big(o_n(g)\big) \tag{37}$$

and by computations in Dym's Fourier Analysis [5], the trace is

$$tr\big(o_n(g)\big) = \frac{\sin\left(\frac{(2n+1)\theta}{2}\right)}{\sin\frac{\theta}{2}} \tag{38}$$

where $\theta$ is the parametrization of $g(\varphi,\theta,\psi)$. Therefore, $\Phi(g)$ is constant on each double cosets, and an explicit expression is given by

$$\Phi(g) = \sum_{n=0}^{\infty}(2n+1)\frac{2}{-n(n+1)}\frac{\sin\left(\frac{(2n+1)\theta}{2}\right)}{\sin\frac{\theta}{2}} \tag{39}$$

Recalling in section 2, one has proved that $(\widehat{f_1 * f_2}) = \widehat{f_2}\widehat{f_1}$. By utilizing this equation,

$$u(g) = \sum_{n=0}^{\infty}(2n+1)tr\left(\frac{2}{-n(n+1)}\hat{f}(n)o_n(g)\right) = (\Phi * f)(g) \tag{40}$$

### *4.2. Applications to the partial differential equations on $S^2$*

A remarkable application of the Fourier analysis on $SO(3)$ is that if one restricts the functions to be constant on the cosets $gK$, where $K = \{h \in SO(3) | h = h(0,0,\psi)\}$, then there is an identity between functions on spheres and functions defined this way. Here, the identity will be denoted together with the identification map $j$ defined by $j(g) = gn = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = x$:

$$f^*(x) = f^*(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = f^*(j(g)) = f(g) = \int_K f(gk)dk \tag{41}$$

One can show that the Laplace-Beltrami operator $\Delta$ defined in this passage acting on each $f^*$ has the similar effect (with a constant $\frac{1}{2}$) as the spherical Laplacian defined below:

$$\Delta_S = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\psi^2} = 2\Delta \tag{42}$$

This is simply because $f$ is constant on the parameter $\psi$, and therefore the terms $-2\frac{\cos\theta}{\sin^2\theta}\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$ disappear. One can therefore utilize the Fourier transform defined in this passage to solve differential equations. A solution to the Poisson's equation $\Delta_S u^*(x) = f^*(x)$, will be computed on $SO(3)$ by considering $2\Delta u(g) = f(g)$. As computed in section 4.1,

$$u(g) = \frac{1}{2}(\Phi * f)(g) = \int_{SO(3)} \sum_{n=0}^{\infty}(2n+1)\frac{1}{-n(n+1)}tr\left(o_n(gh^{-1})\right)f(h)dh \quad (43)$$

therefore, if one sets $x = j(g)$, and $y = j(h(\varphi,\theta,0))$,

$$u^*(x) = u(g)\frac{1}{4\pi}\int_{S^2}\left(\int_K \sum_{n=0}^{\infty}(2n+1)\frac{1}{-n(n+1)}tr\left(o_n(gk^{-1}h^{-1})\right)\right)dk\Big)f^*(y)d\sigma(y) \quad (44)$$

Here, one may compute $\int_K \sum_{n=0}^{\infty}(2n+1)\frac{1}{-n(n+1)}tr\left(o_n(gk^{-1}h^{-1})\right)\right)dk$ directly, since by normal parametrization,

$$g = h(\varphi_1, \theta_{1,0}), \qquad k = h(0,0,\psi), \qquad h^{-1} = h(-\psi_1, -\theta, -\varphi) \quad (45)$$

and the integration $\int_K f(gk)dk = \int_K f(gk'k)dk$, for $k' \in K$. Through the matrix multiplication, one gets the parametrization $\tilde{\theta} = \arccos(\cos\theta_1\cos\theta + \cos\psi\sin\theta_1\sin\theta)$. One may therefore get the Green's function

$$\Phi^*(x(\varphi_1,\theta_1), y(\varphi,\theta)) = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty} -\frac{(2n+1)}{n(n+1)}\frac{\sin\left(\frac{(2n+1)\arccos(\cos\theta_1\cos\theta+\cos\psi\sin\theta_1\sin\theta)}{2}\right)}{\sin\frac{\arccos(\cos\theta_1\cos\theta+\cos\psi\sin\theta_1\sin\theta)}{2}}d\psi \quad (46)$$

and the solution is given by $u^*(x) = \frac{1}{4\pi}\int_{S^2}\Phi^*(x,y)f^*(y)d\sigma(y)$. This seems to be the best one can do.

## 5. Limitations & prospects

The parametrization in this passage is fixed to be the Euler angle of a rotation matrix. However, this parametrization is not perfect because it's not bijective when $\theta = 0$, since in this case the matrix would look like

$$\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\cos(\varphi+\psi) & -\sin(\varphi+\psi) & 0\\ \sin(\varphi+\psi) & \cos(\varphi+\psi) & 0\\ 0 & 0 & 1\end{bmatrix} \quad (47)$$

In this case, one cannot get the corresponding Euler angle using formula $(\varphi,\theta,\psi) = (\arctan\frac{h_{23}}{h_{13}}, \arccos h_{33}, \arctan(-\frac{h_{32}}{h_{31}}))$, from which this study has derived the Laplace-Beltrami operator. From this point of view, a more reasonable parametrization is to use quaternions. However, the symmetry that the Euler angle possesses may not appear in the case of quaternions. The fact that the Laplace-Beltrami operator defined in this passage commutes with left- and right-shift operators still need to check the convergence of $\left(\sum_{n=0}^{\infty}\frac{1}{n!}t^n\partial_i^n\right)$, and the validity of interchanging limit operator and Laplace-Beltrami operator. Solutions to the differential equations given in this passage are based on the assumption that the solution actually exists, so the existence of such solutions remains to be proved. It is totally probable that one can find other expansions for different differential operators, just like when handling second-order elliptic equations in Euclidean spaces. Indeed, throughout the texts, the most discovers relative to the Laplace-Beltrami operator is left- and right-shift invariant, and the fact that Legendre polynomial is an eigenfunction of the Laplace-Beltrami operator defined this way in the space of double cosets. If a new differential operator is in consideration, one may also find an eigenfunction in $K/SO(3)/K$, and one only needs this differential operator to be commutative with left- and right-shift operators on $K/SO(3)/K$, rather than the whole $SO(3)$, to make every $o_n^{ij}$ constructed similarly be the eigenfunction of this differential operator. Therefore, more solutions to the differential equations on $SO(3)$ will also appear to give a solution to the associative differential equations on $S^2$. In general, this study is only an explicit analysis on $SO(3)$, and doesn't give any generalization of explicit representations of higher dimensional Lie groups, or arbitrary manifolds. Indeed, one needs very strong symmetries, such as in this passage, that the space of double cosets is commutative, and the Laplace-Beltrami operator would be the same if one interchanges $\varphi$ and $\psi$.

## 6. Conclusion

In summary, this study focused on the explicit expression of Laplace-Beltrami operator on SO(3) and its relation with Fourier analysis, which have given a fruitful result in solving differential equations. first noticed that using Fourier expansion, the algebra of convolutions is converted into the algebra of

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multiplications, which gives the fundamental solutions to the differential equations. Then, Laplace-Beltrami operator was defined with the help of Lie groups. the analysis of the Laplace-Beltrami operator showed its left- and right-shift invariant properties, resulting in the deduction that $o_n^{ij}$ are eigenfunctions of the Laplace-Beltrami operator. There are still limitations considering the parametrization is not bijective, and the analysis in this passage is not general enough to cover all linear or semi-linear second-order differential equations. Nevertheless, it is still hopeful to construct more general Fourier analysis as a machinery to solve more differential equations, and even give them explicit solutions. This is still a long way to go, but one may find some instructive analysis, including the construction of the Laplace-Beltrami operator in this study, and continue to provide deeper results.

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