

# Composing group action to permutation representation

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**Abstract.** A mapping that satisfies two specific axioms provides a common notion of group action. A homomorphism translating from a group to a symmetric group of a certain set can also be used to describe group action. Therefore, any example of the group actions can be stated based on the second equivalent definition, such as the regular action, natural matrix action, coset action, and  $\mathbb{Z}^2$  acting on  $\mathbb{R}^2$ , etc. It is necessary to examine the concepts of the orbit and stabilizer of a group in order to reveal the orbit-stabilizer theorem. After the preparatory work, the orbit-stabilizer theorem can be proved by defining a mapping from the orbit to the stabilizer and then checking that the mapping is well-defined and bijective. To derive Burnside's lemma, it needs to introduce the set of fixed points which is related to the concept of the stabilizer. Through the orbit-stabilizer theorem along with the fact that a set is a disjoint union of orbits, Burnside's lemma can be confirmed. Moreover, it is natural to compose a group action with a linear representation, and then a representation would be obtained, which is permutation representation. Further, one must calculate the character of the permutation representation, the dimension of the fixed subspace, and the dimension of  $\mathbb{C}X^G$ . Then it can show Burnside's lemma in another way by permutation representation.

**Keywords:** Group Action, Orbit-Stabilizer Theorem, Burnside's Lemma, Permutation Representation.

## 1. Introduction

Firstly, this article reviews the concept of group action. For a set  $X$  and a group  $G$ , the group action is often defined as [1]: an action of  $G$  on  $X$  is a mapping  $\rho: G \times X \rightarrow X$  satisfying two properties: (i)  $\rho(1_G, x) = x$  for each  $x \in X$ ; (ii)  $\rho(gg', x) = \rho(g, \rho(g', x))$  for all  $g, g' \in G$  and  $x \in X$ .

But some other works of literature, define the group action as follows [2], which is distinguished from [1]: an action of  $G$  on  $X$  is a mapping:

$$\rho: G \rightarrow \text{Sym}(X) \quad (1)$$

which is homomorphic. Then this article will show that those two definitions of group actions are equivalent, and then it will list several typical examples to review the concept of group action. Meanwhile, this section will introduce the definition of transitive group actions to derive the following results.

In the second section, this article will briefly introduce the notions of an orbit and a stabilizer. Due to the similarities between the definitions of orbits and stabilizers, it is natural to assert that they are

related closely. Then this article states and proves the orbit-stabilizer theorem by using the group action method as [3].

Since the stabilizer has been defined before, the set of fixed points which has a similar form to a stabilizer can be defined naturally. Then this article in the third section states and shows Burnside's lemma which gives the relationship between orbits and sets of fixed points by analogy to orbit-stabilizer theorem. Such a lemma states that the average of the set of fixed points determines how many orbits a group has on a set [4]. Orbit-stabilizer theorem relates closely to the result and it plays a key role in the proof of Burnside's Lemma [5].

Moreover, this article reviews the concept of representation; composing a group action of  $G$  on  $X$  and the linear representation of  $\text{Sym}(X)$  on  $\text{GL}_n(\mathbb{C})$ , the concept of permutation representation can be derived by [6]. For a permutation representation

$$\tilde{\rho}: G \rightarrow \text{GL}(\mathbb{C}X),$$

it needs to consider the set  $\mathbb{C}X = \{\sum_{x \in X} c_x x \mid c_x \in \mathbb{C}\}$ .

A significant fact is that  $\mathbb{C}X$  is a vector space with a finite set  $X$  as the basis, and it is an important and difficult point in the following proofs. Having defined the set of fixed points earlier, the article then gives the following definition: fixed subspace. Then this article will compute the character of permutation representations, the dimension of fixed subspace, and the dimension of  $\mathbb{C}X^G$ . Therefore, by the above three results of computation, it follows Burnside's lemma in another way by an equation in [6].

To compute the character of permutation representations, it needs to determine the matrix of  $\tilde{\rho}$  with respect to a basis  $X$ , where  $\tilde{\rho}$  is a permutation representation associated to  $\rho$  [7, 8]. To compute the dimension of fixed subspaces, it needs to write  $V = \bigoplus_{i=1}^r n_i V_i$  where any  $V_i$  is an irreducible  $G$ -invariant subspace, and all these  $V_i$ 's sub-representations stretch through the classes of group  $G$  representations that are irreducible. Suppose that  $V_1$  is equivalent to a trivial representation. To compute the dimension of  $\mathbb{C}X^G$ , it needs to set the orbit of  $G$  on  $X$  as the form of disjoint union of orbits  $\mathcal{O}_i$  and  $v_i = \sum_{x \in \mathcal{O}_i} x$ . Then it follows that  $v_1, \dots, v_n$  span  $\mathbb{C}X^G$ . After above preparation, Burnside's lemma can be proved in the other way.

## 2. Methods

### 2.1. Regarding $\rho(g, x)$ as $\rho_g(x)$

Fixing a set  $X$  and a group  $G$ .

**Definition 1.** A function  $\rho$  is used to specify a  $G$  action on  $X$ :

$$\rho: G \times X \rightarrow X$$

satisfying the next two criteria: (i)  $\rho(1_G, x) = x$  for every  $x \in X$ ; (ii)  $\rho(gg', x) = \rho(g, \rho(g', x))$  for every  $g, g' \in G$  and  $x \in X$ . One often denotes the action by  $\rho(g, x) = g \cdot x$ . Then the properties above become: (i')  $1_G \cdot x = x$  for every  $x \in X$ ; (ii')  $gg' \cdot x = g \cdot (g' \cdot x)$  for every  $g, g' \in G$  and  $x \in X$ .

In the second section, this article will briefly introduce the notions of an orbit and a stabilizer. Due to the similarities between the definitions of orbits and stabilizers, it is natural to assert that they are related closely. Then this article states and proves the orbit-stabilizer theorem by using the group action method.

Seeing that the form of the definition of group actions in most literature, the method to show both two definitions are equivalent is to regard  $\rho(g, x)$  as  $\rho_g(x)$ . Then  $\rho_g$  is an element of  $\text{Sym}(X)$ . Therefore, it needs to check that  $\rho_g$  is a homomorphism and satisfies the properties of group actions. The comparable definition of group action is as follows.

**Definition 2.** The action of  $G$  on  $X$  is homomorphism as (1).

After showing that group actions can be defined in two ways and that they are equivalent, this article will list several typical examples. These examples are the regular action, natural matrix action,

coset action, and  $\mathbb{Z}^2$  acting on  $\mathbb{R}^2$  respectively. Then the article will check the fact those two ways of defining a group action are the same. Meanwhile, this article will introduce several concepts by the way, such as transitivity, etc., which will play a role in the subsequent arguments.

### 2.2. Proof of Orbit-Stabilizer theorem by group action

Before stating the orbit-stabilizer theorem, this article reviews the concept of  $G$ -invariant. A subset  $Y$  of  $X$  is named  $G$ -invariant if  $\rho_g(y) \in Y$  for all  $g \in G, y \in Y$ . One idea is to divide a set into the disjoint unions of invariant subsets. Then it obtains the following two definitions.

**Definition 3.** For the group action

$$\rho: G \rightarrow \text{Sym}(X),$$

an orbit of an element  $x$  of the set  $X$  under  $G$  is defined as

$$G \cdot x = \{\rho_g(x) \mid g \in G\}.$$

**Definition 4.** For a transitive group action

$$\rho: G \rightarrow \text{Sym}(X)$$

where  $|X| \geq 2$  and a fixed element  $x$  of the set  $X$ , the subgroup of  $G$

$$G_x = \{g \in G \mid \rho_g(x) = x\}$$

is named the stabilizer of  $x$ .

For the orbit-stabilizer theorem, the article proves it by defining a mapping:

$$\varphi: G \cdot x \rightarrow G/G_x.$$

Then it needs to check that  $\varphi$  is surjective and injective respectively. Then it follows the orbit-stabilizer theorem in the case that  $G$  is finite.

### 2.3. Proof of Burnside's lemma by permutation representation

The stabilizer this paper has defined earlier. It is natural to define the set of fixed points.

**Definition 5.** For a group action:

$$\rho: G \rightarrow \text{Sym}(X)$$

and a fixed  $g \in G$ ,

$$X^g = \{x \in X \mid \rho_g(x) = x\}$$

is named the set of fixed points of  $g$ .

The first strategy in this article is based on the discovery that  $X$  is a disjoint union of orbits and the orbit-stabilizer theorem. For the second method, having a group action  $\rho: G \rightarrow \text{Sym}(X)$  in view, it might be put together using a linear representation  $\varphi: S_n \rightarrow \text{GL}_n(\mathbb{C})$  if  $X$  is a finite set whose order is  $n$ . A finite group's linear representation is given by the homomorphism  $\rho$  from  $G$  to  $\text{GL}(V)$  [9]. Then a representation of  $G$  would be obtained.

**Definition 6.** For a group action  $\rho: G \rightarrow \text{Sym}(X)$ , the homomorphism  $\tilde{\rho}: G \rightarrow \text{GL}(CX)$  is called the permutation representation associated to  $\rho$  if

$$\tilde{\rho}_g \left( \sum_{x \in X} c_x x \right) = \sum_{x \in X} c_x \rho_g(x).$$

The set

$$\mathbb{C}X = \left\{ \sum_{x \in X} c_x x \mid c_x \in \mathbb{C} \right\}$$

in Definition 6 is a vector space with finite set  $X$  as a basis. It is easy to check that  $\tilde{\rho}$  is a linear extension of  $\rho_g$  on the basis  $X$  to  $\mathbb{C}X$ . Having defined the set of fixed points earlier, this article then gives the following definition of the fixed subspace.

**Definition 7.** For a linear representation  $\varphi: G \rightarrow \text{GL}(V)$ , the fixed subspace of  $G$  is defined as the set:

$$V^G = \{v \in V \mid \varphi_g(v) = v, \forall g \in G\}.$$

A fixed subspace of  $G$  is clearly  $G$ -invariant. Then it needs to compute the character of the permutation representation  $\tilde{\rho}$ , the dimension of the fixed subspace  $V^G$ , and the dimension of  $\mathbb{C}X^G$ . Once the preparatory work has been done, Burnside's lemma can be proved in the other way.

### 3. Results and Discussion

#### 3.1. Equivalent definition and examples of the group action

Fixing a set  $X$  and a group  $G$ . For each  $g \in G$ , define that:

$$\rho_g: X \rightarrow X, \quad x \mapsto g \cdot x$$

which is a bijection from  $X$  to  $X$ , i.e., an element of  $\text{Sym}(X)$ . Define the mapping:

$$\rho: G \rightarrow \text{Sym}(X), \quad g \mapsto \rho_g.$$

For any  $x \in X$ , the second property (ii) of the group action follows that:

$$\rho(gg')(x) = gg' \cdot x = g \cdot (g' \cdot x) = g \cdot (\rho_{g'}(x)) = \rho_g(\rho_{g'}(x)) = (\rho(g) \circ \rho(g'))(x)$$

Therefore, it has been checked that  $\rho(gg') = \rho(g) \circ \rho(g')$ , i.e., the mapping  $\rho$  is homomorphic. It is also easy to check that the homomorphism  $\rho$  defined in Definition 2 satisfies both properties in Definition 1. Thus, Definition 1 and Definition 2 are equivalent.

Then, in order to comprehend and compare the equivalent definition of the group action, it is required to provide several typical instances.

**Example 1 (Regular action).** Let  $G$  be a group. Define a mapping:

$$\sigma: G \rightarrow \text{Sym}(G), \quad g \mapsto \sigma_g$$

where  $\sigma_g(x) = gx$ . The mapping  $\sigma$  defined above is named a regular action of  $G$  on  $G$ . Regular action, one of the simplest types of group action, is essential to Cayley's theorem's verification.

**Theorem 1 (Cayley's theorem).** Any group is able to be isomorphic to a set's permutation group.

**Proof.** Define the mapping  $\sigma$  and  $\sigma_g$  as in Example 1. Then  $\sigma$  maps  $G$  to the symmetric group of  $G$ . For any  $\sigma_{g_1}, \sigma_{g_2} \in \sigma(G)$ ,  $x \in G$ , since:

$$\sigma_{g_1} \sigma_{g_2}^{-1}(x) = \sigma_{g_1} \sigma_{g_2^{-1}}(x) = g_1 g_2^{-1} x = \sigma_{g_1 g_2^{-1}}(x),$$

It follows that  $\sigma_{g_1} \sigma_{g_2}^{-1} = \sigma_{g_1 g_2^{-1}} \in \sigma(G)$ , i.e.,  $\sigma(G) < \text{Sym}(G)$ . Then  $G$  is isomorphic to  $\sigma(G)$ .

**Example 2 (Natural matrix action).** Define the general linear group  $\text{GL}_n(\mathbb{Z})$  acting on  $\mathbb{Z}^n$  as:

$$g: \text{GL}_n(\mathbb{Z}) \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n, \quad A \times x \mapsto Ax,$$

where  $\mathbb{Z}^n$  can be regarded as the set of  $n \times 1$  column vectors.

The group action defined in Example 2 can be also state as:

$$g: \text{GL}_n(\mathbb{Z}) \rightarrow \text{Sym}(\mathbb{Z}^n), \quad A \mapsto g_A$$

with  $g_A(x) = Ax$  where  $x \in \mathbb{Z}^n$  as Definition 2.

**Example 3 (Coset action).** For a group  $G$  and its subgroup  $H < G$ , the action

$$\tau: G \rightarrow \text{Sym}(G/H)$$

given by  $\tau_g(xH) = gxH$  is named a coset action of  $G$  on  $G/H$ .

A group action  $\rho: G \mapsto \text{Sym}(X)$  is called to be transitive if for both  $x, y \in X$ , there exists a  $g \in G$  satisfying  $\rho_g(x) = y$ . Now choose  $\bar{x}, \bar{y} \in G/H$ , where  $\bar{x} = xH, \bar{y} = yH$ . For  $x, y \in G$ , there must exist a  $g \in G$  satisfying  $gx = y$ . Therefore,

$$\tau_g(\bar{x}) = \tau_g(xH) = gxH = yH = \bar{y}.$$

Thus, the action  $\tau$  in Example 3 (Coset action) is transitive.

**Example 4 ( $\mathbb{Z}^2$  acting on  $\mathbb{R}^2$ ).** The group  $\mathbb{Z}^2$  acting on  $\mathbb{R}^2$  can be defined as:

$$f: \mathbb{Z}^2 \times \mathbb{R}^2 \rightarrow, (a, b) \times (x, y) \mapsto (a + x, b + y).$$

It is straightforward to verify that  $f$  satisfies the properties in Definition 1. It can be also defined as:

$$f: \mathbb{Z}^2 \rightarrow \text{Sym}(\mathbb{R}^2), (a, b) \mapsto f_{(a,b)}$$

with  $f_{(a,b)}(x, y) = (a + x, b + y)$  where  $(x, y) \in \mathbb{R}^2$  as Definition 2.

### 3.2. Orbit-Stabilizer theorem

Due to the similarities between the definitions of orbits and stabilizers, it can assert that they are related closely. Then it derives the following theorem.

**Theorem 2 (Orbit-stabilizer theorem).** For a finite set  $X$  under the action of the finite group  $G$ , it has

$$|G \cdot x| = \frac{|G|}{|G_x|} \quad (2)$$

for any  $x \in X$ .

**Proof.** For a fixed  $x \in X$ , define the function:

$$\varphi: G \cdot x \rightarrow G/G_x, \rho_g(x) \mapsto gG_x.$$

Suppose  $\rho_g(x) = \rho_h(x)$  for some  $g, h \in G$ . Then

$$\rho_h^{-1}\rho_g(x) = \rho_h^{-1}\rho_h(x).$$

Further,

$$\rho_{h^{-1}g}(x) = \rho_{h^{-1}h}(x) = \rho_{I_G}(x) = x.$$

Thus  $h^{-1}g \in G_x$ . Then  $\varphi$  is well-defined since  $gG_x = hG_x$  by left cosets partitioning groups.

Let  $\varphi(\rho_{g_1}(x)) = \varphi(\rho_{g_2}(x))$  for some  $g_1, g_2 \in G$ . Then  $g_1G_x = g_2G_x$  and  $g_2^{-1}g_1 \in G_x$ . So  $\varphi$  is injective since  $x = \rho_{g_2^{-1}g_1}(x)$  and

$$\rho_{g_2}(x) = \rho_{g_2}(\rho_{g_2^{-1}g_1}(x)) = \rho_{g_2g_2^{-1}g_1}(x) = \rho_{g_1}(x).$$

By the definition of  $\varphi$ ,

$$\varphi(\rho_g(x)) = gG_x,$$

so  $\varphi$  is surjective. Therefore,  $\varphi$  is bijective and since  $X$  is a finite set, it follows (2) for any  $x \in X$ .

In fact, it can be done to expand the orbit-stabilizer theorem to include more situations, that is,  $G$  is any group, see [10]. The idea to show the general case is consistent to the above proof of Theorem 2.

### 3.3. Burnside's lemma

Having defined the set of fixed point, this article gives the following theorem by analogy to orbit-stabilizer theorem. This theorem gives the relationship between orbits and sets of fixed points.

**Theorem 3 (Burnside's lemma).** For a group action  $\rho: G \rightarrow \text{Sym}(X)$ , let  $\mathcal{O}$  be the set of orbits of  $G$  on  $X$ . It has

$$|\mathcal{O}| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Burnside's lemma clarifies that the quantity of a group's orbits on a set is the average of the set fixed points. Orbit-stabilizer theorem relates closely to the above theorem (Burnside's lemma).

**Proof.** It needs to exploit the orbit-stabilizer theorem along with the fact that a set is a disjoint union of orbits, i.e.,

$$\begin{aligned} \sum_{g \in G} |X^g| &= |\{(g, x) \in G \times X \mid g \cdot x = x\}| = \sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|G \cdot x|} \\ &= |G| \sum_{\mathcal{O}_i \in \mathcal{O}} \sum_{x \in \mathcal{O}_i} \frac{1}{|\mathcal{O}_i|} = |G| \sum_{\mathcal{O}_i \in \mathcal{O}} 1 = |G| |\mathcal{O}|. \end{aligned}$$

For the group action  $\rho: G \rightarrow S_n$ , a natural idea is to compose this group action  $\rho$  with a linear representation  $\varphi: S_n \rightarrow \text{GL}_n(\mathbb{C})$  and then a new representation of  $G$  would be obtained. This motivates the concept of permutation representation as Definition 6. Hence it needs to compute the character of the permutation representation  $\tilde{\rho}$ , the dimension of fixed subspace  $V^G$ , and the dimension of  $\mathbb{C}X^G$ , etc.

**Lemma 1 (Character of permutation representation).** For a group action  $\rho: G \rightarrow \text{Sym}(X)$ , it has

$$\chi_{\tilde{\rho}}(g) = |X^g|.$$

**Proof.** Let  $[\tilde{\rho}_g]$  be the matrix of  $\tilde{\rho}$  with respect to the basis  $X$ , and set  $X = \{x_1, \dots, x_n\}$ . Then

$$[\tilde{\rho}_g]_{ij} = \begin{cases} 1, & x_i = \rho_g(x_j); \\ 0, & \text{else.} \end{cases} \quad (3)$$

It follows the form of this matrix. Thus  $\chi_{\tilde{\rho}}(g) = \text{tr}([\tilde{\rho}_g]) = |X^g|$ .

**Lemma 2 (Dimension of fixed subspace).** For a linear representation  $\varphi: G \rightarrow \text{GL}(V)$ , let  $\chi_1$  be a trivial character of  $G$ . It has

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \overline{\chi_1(g)}.$$

**Proof.** Without loss of generality, writing  $V = \bigoplus_{i=1}^r n_i V_i$  where any  $V_i$  is an irreducible  $G$ -invariant subspace. Then all of  $V_i$ 's sub-representations stretch through the equivalent classes of irreducible representations of  $G$ . Suppose that  $V_1$  is equivalent to a trivial representation. Let  $v = \sum_{i=1}^r v_i$  with  $v_i \in n_i V_i$ , and  $\varphi_g v = \sum_{i=1}^r (n_i \varphi|_{V_i}) g v_i = v_1 + \sum_{i=2}^r (n_i \varphi|_{V_i}) g v_i$ . Hence  $g \in V^G$  if and only if  $v_i \in n_i V_i^G$  for any  $2 \leq i \leq r$ , i.e.,

$$V^G = n_1 V_1 \oplus \bigoplus_{i=2}^s n_i V_i^G.$$

Since  $V_i^G$ 's is  $G$ -invariant and  $V_i$ 's are not equivalent to the trivial representation for any  $i \geq 2$ , it has  $V_i^G = 0$ . Then  $V^G = n_1 V_1$ . Hence it obtains the equation (3).

Having stated Lemma 1 and Lemma 2, it can compute the dimension of  $\mathbb{C}X^G$  with a permutation representation.

**Lemma 3 (Dimension of  $\mathbb{C}X^G$ ).** Let  $\rho: G \rightarrow \text{Sym}(X)$  be a group action. Then

$$\dim \mathbb{C}X^G = |\mathcal{O}|.$$

**Proof.** Set the orbit of  $G$  on  $X$  as  $\mathcal{O} = \{\mathcal{O}_1, \dots, \mathcal{O}_n\}$  and  $v_i = \sum_{x \in \mathcal{O}_i} x$ . Let  $y = \rho_g(x)$  and  $\tilde{\rho}$  be the permutation representation associate to  $\rho$ . Then

$$\tilde{\rho}_g(v_i) = \sum_{x \in \mathcal{O}_i} \rho_g(x) = \sum_{y \in \mathcal{O}_i} y = v_i.$$

So  $v_1, \dots, v_n \in \mathbb{C}X^G$ . Since  $X$  is a disjoint union of orbits,

$$\langle v_i, v_j \rangle = \begin{cases} |\mathcal{O}_i|, & i = j, \\ 0, & \text{else.} \end{cases}$$

Thus  $\{v_1, \dots, v_n\}$  is an orthogonal set of non-zero vector. Let

$$v = \sum_{x \in X} c_x x \in \mathbb{C}X^G.$$

Suppose  $z = \rho_g(y) \in G \cdot y$ . It has

$$v = \sum_{x \in X} c_x x = \tilde{\rho}_g v = \sum_{x \in X} c_x \rho_g(x) = \sum_{x \in X} c_x y,$$

and then

$$\rho_g(v) = \sum_{x \in X} c_x \rho_g(x) = \sum_{x \in X} c_x y = \sum_{x \in X} c_x z.$$

Hence  $c_y = c_z$ . Then there exists  $i \in \mathbb{N}$  for all  $1 \leq i \leq n$  such that  $c_x = c_i$  for all  $x \in \mathcal{O}_i$ . Therefore,

$$v = \sum_{x \in X} c_x x = \sum_{i=1}^n \sum_{x \in \mathcal{O}_i} c_x x = \sum_{i=1}^n c_i \sum_{x \in \mathcal{O}_i} x = \sum_{i=1}^n c_i v_i \quad [6].$$

Thus  $v_1, \dots, v_n$  span  $\mathbb{C}X^G$ .

Now this article can use another method of permutation representation to prove Burnside's lemma.

**Proof (Another proof of Burnside's lemma).** By above three lemmas, i.e., Lemma 1-3, it has

$$|\mathcal{O}| = \frac{1}{|G|} \sum_{g \in G} \chi_{\tilde{\rho}}(g) \overline{\chi_{\tilde{\rho}}(g)} = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

In fact, Theorem 3 (sometimes known as Burnside's lemma) was really discovered by Frobenius in 1887, who Burnside only acknowledges in his book [4].

#### 4. Conclusion

The standard definition of a group action is given by a mapping satisfying two axioms, which is equivalent to another definition that is defined as a homomorphism which is from a group to a symmetric group of a certain set equivalently. The examples of group actions, stated based on the first definition, such as the regular action, natural matrix action, coset action, and  $\mathbb{Z}^2$  acting on  $\mathbb{R}^2$ , etc, can be restated as the homomorphisms satisfying the second definition equivalently. In particular, the regular action defined as the second way plays a key role on showing Cayley's theorem.

It is necessary to examine the concepts of the orbit and stabilizer of a group in order to understand the orbit-stabilizer theorem. After the preparation, the orbit-stabilizer theorem can be proved by defining a mapping  $\varphi$  from the orbit  $G \cdot x$  to the stabilizer  $G/G_x$  and then checking that the mapping  $\varphi$  is well defined and bijective. Thus, it follows the relation between the orbit and the stabilizer.

Moreover, the general case of the orbit-stabilizer theorem, that  $G$  is any group can be reached, whose verification is consistent to the above proof of the finite case.

For Burnside's lemma, deriving the set of fixed points is necessary, which is connected to the concept of the stabilizer. Then Burnside's lemma stating that the average of the set of fixed points determines how many orbits a group has on a set can be proved through the orbit-stabilizer theorem. On the other hand, it is natural is to compose a group action  $\rho: G \rightarrow \text{Sym}(X)$  with a linear representation  $\varphi: \mathcal{S}_n \rightarrow \text{GL}_n(\mathbb{C})$  where the order of the finite set  $X$  is  $n$ , and then a representation of would be obtained, which is named permutation representation. Furthermore, it needs to compute the character of the permutation representation  $\tilde{\rho}$ , the dimension of the fixed subspace  $V^G$ , and the dimension of  $\mathbb{C}X^G$ . Then it can show Burnside's lemma in another way by permutation representation.

### References

- [1] Dummit D S and Foote R M 2004 Group Actions. in *Abstract Algebra (3rd ed.)*, New York, Wiley, 41.
- [2] Eie M and Chang S T 2010 A Course on Abstract Algebra. *Iowa, World Scientific*, 253.
- [3] Hasselbarth W 1986 An Invitation to Permutation Representations of Groups. *Croatica Chemica Acta*, 59(3) 565-582.
- [4] Burnside W 2017 Theory of groups of finite order (2nd ed). *New York: Dover Publications, Inc.Mineola*.
- [5] Smith J D 2006 Permutation representations of left quasigroups. *Algebra universalis*, 55 387-406.
- [6] Steinberg B 2011 Permutation Representation. in *Representation Theory of Finite Groups: An Introductory Approach*, Netherlands, Springer Nature, 85-88.
- [7] Cherniavsky Y and Sklarz M 2008 Conjugacy in permutation representations of the symmetric group. *Communications in Algebra*, 1726-1738.
- [8] Cherniavsky Y and Bagno E 2006 Permutation representations on invertible matrices. *Linear algebra and its applications*, 494-518.
- [9] Serre J P 1977 Generalities on linear representation. *Linear representation*, New York, Springer.
- [10] Artin M 2011 The Operation on Cosets. in *Algebra (2nd ed.)*, Boston, Pearson Prentice Hall, 179.