DOI: 10.54254/2753-8818/14/20240891

The concept of infinity and the development of set theory solutions

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Abstract. The concept of infinity is intricately connected to and comprehended via the framework of cardinal and ordinal numbers. Cardinal and ordinal numbers are fundamental mathematical ideas within the field of set theory. The cardinal number is used to denote the quantity of items inside a given set while ordinal defines basic algorithms. This article demonstrates how the discipline of set theory may be used as a tool to investigate the nature of infinity, or at the very least give some insights into the subject. The idea of infinity may be better understood by looking at it through the lens of set theory and the commutative property. In addition to that, this research presents the connection and compatibility examination of Cohen's operation upon the Zermelo Freankel axiomatic framework. These accomplishments are discussed in this article as illuminating insights for readers to consider while imagining the ultimate solution to the problems posed by the Continuum Hypothesis and the nature of infinity. Through the use of examples from contemporary researchers, the multiverse and indeterminism are introduced as potential approaches to the problem of how to solve Cantor's legacy in the future.

Keywords: Continuum Hypothesis, Infinity, Set Theory.

1. Introduction

The purpose of research is to present how set theory can be used as a tool to find out, or at least to provide insights into the answer to the nature of infinity. In light of this, the study will divide its conversation into three distinct parts: to begin, it will provide a definition of infinite and explain the mathematical concepts that will be used throughout the remainder of the essay. In the next paragraph, it will discuss many hypotheses on the nature of the infinite. The last part of this article presents an argument that counters the idea that infinity is only an extension or continuation of the natural numbers. As a direct consequence of this discussion, set theory and its development will be subjected to scrutiny.

Infinity does not make up a component of any of the number systems that come to mind when we think of numbers [1]. This is true for natural numbers, integers, rational numbers, real numbers, and ordinal numbers. The concept of infinity ought to be captured by the combination of cardinal and ordinal numbers, or by operations on those numbers. However, one cannot get a comprehensive understanding of the way mathematicians think about infinity by concentrating just on the issues mentioned here. The development of set theory is something that should be taken into consideration in this issue.

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2. Infinity in the light of modern mathematical methods

The essence of infinity is both related and understood in the sense of cardinal and ordinal numbers. Both cardinal and ordinal numbers are mathematical concepts in set theory. Where the cardinal number is used to represent the number of elements in a set, such as for the set $A = \{a, b\}$, The cardinal number of the set A is 2. An ordinal is used to denote the position or order of a thing in an ordered set, e.g., for an element a in a set A, its ordinal is 1. In the following paragraph, this study will use the thought experiment of the Hilbert Hotel, proposed by the German mathematician David Hilbert, which, in its most basic version, states that in a hotel with infinitely many rooms, one can have guests in room n move to room n+1 so that room 1 will be available for new guests. In a hotel with an infinite number of rooms, one can have the guest in room n move to room n+1 so that room 1 is available to the new guest. In the next step, this research will also use ordinal arithmetic: ordinal arithmetic is a mathematical operation performed on the set of ordinal numbers, which extends the operations on natural numbers and cardinal numbers. In ordinal arithmetic, given two ordinals a and b, their sum, a + b, means that the ordinal b is placed after the ordinal a to form a new ordered set. The fact that this process is ordered means that placing a first and then b, or b first and then a results in different outcomes.

The essence of infinity is the base of the infinite set, and for different infinite sets, there are different grades of infinity, which could be described as different cardinalities [2, 3]. The natural numbers are the smallest set of infinities whose base is labeled $\aleph_0$. Natural numbers are the lowest-ranking set of infinities because every infinite subset of the natural numbers can correspond one-to-one with the set of natural numbers in some way. This discussion will use the tools of the Hilbert Hotel thought experiment to show that, for a hotel with only one floor, no matter how the existing rooms are arranged, no matter if they are vacant or not, one can make the rooms There are no vacancies in the room.

Based on the continuum hypothesis, the real numbers are the second smallest infinite set whose base is labeled as $\aleph_1$. The real numbers "continuously" cover the whole axis, whereas the natural numbers leave gaps on the axis so that a one-to-one correspondence between the real and natural number points on the axis reveals that there is a surplus of natural number points, which represents the fact that the base of the real numbers is larger than the base of the natural numbers. At the same time, one can construct a higher level of infinity by using one level of infinity as the exponent and 2 as the base:

$$2^{\aleph_0} = \aleph_{1,2}^{\aleph_1} = \aleph_{2,2}^{\aleph_2} = \aleph_3 \ldots \tag{1}$$

Thus for any $\aleph_n, n \in N, n \neq 0$ are bases of different infinite sets and infinities of different grades. The essence of infinity relates to ordinal numbers. In the Hilbert Hotel problem, for a new guest, we moved each of the already-occupied guests back to one room. But in the process, we ignored the trouble caused to the existing roomers. Let's consider another case, if one guest comes first and an infinite number of guests come after, then all of them don't need to change rooms. So we know that "1 plus infinity is still infinity" will not cause any trouble for the existing guests, while "infinity plus 1 is still infinity" will cause trouble, so we can write the following inequality.

$$\infty + 1 \neq 1 + \infty \tag{2}$$

Therefore in the case of infinite ordinals, the law of exchange for addition is not valid. Another example helps to show that the law of exchange for addition is not valid in the case of infinite ordinals. This exchange is defined as a commutative property, according to the studies of Abelian groups.

Since the consideration of order was introduced in Hilbert's Inn, it is necessary to consider the problem of queues [4, 5]. Suppose one is giving out red tickets, each of which corresponds to a natural number, as a way of marking the order in which people are in the queue. If one uses up all the red tickets, because an infinite number of guests have already arrived, and at that point if a new guest arrives, one could have each guest move back to one place and take the red #1 ticket to the newly arrived guest. But this newly arrived guest then cuts in line with everyone else, which is very unfair. At

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this point, the only fair way to do this is to give this newly arrived guest the blue #1 ticket and have the blue line up after the red, from which one can write two equations:

$$\infty + 1 > \infty \text{ when using the blue ticket; } 1 + \infty = \infty \text{ when using the red} \tag{3}$$

Thus it can be learned that adding something to the left of infinity is not the same as adding something to the right of infinity, thus proving once again that commutative property for addition does not exist in the case of infinite ordinals. The nature of infinity is understood as ordinal, and the operation of infinity requires the use of ordinal arithmetic. The natural numbers are the set of non-negative integers and satisfy the exchange law of addition. It follows from the second part of the argument that in the case of infinite ordinals, the exchange law of addition does not exist for semigroups, so infinity is not a natural number. Since the extensions of the natural numbers: integers, rational numbers, real numbers, and imaginary numbers all satisfy the exchange law of addition, ∞ is not an extension of any natural number either.

To summarize, a fully developed concept of infinity does not belong to the natural numbers, the integers, the rational numbers, the real numbers, or the ordinal numbers. the essence of infinity should be a concept defined by the cardinal and ordinal numbers or operations together. Meanwhile, an examination in an elementary algebraic field like radicals and ordinals does not present a complete picture of thinking about infinity in the field of mathematics. To discuss it effectively, the development of set theory must be taken into account.

3. The Infinity Problem in the Set Theory Context

3.1. History of the continuum hypothesis

Cantor first introduced the idea of set theory in a letter to his friend R. Dedekind on December 7, 1873, which has been designated as the day set theory was born. In the following 10 years, Cantor published a series of papers and created a new branch of mathematics, set theory, which, together with the rigorous theory of limits by A. L. Cauchy and others, and the rigorous theory of real numbers by R. Dedekind and others, gave the whole of mathematics a complete system and a reliable foundation [6]. However, only two years later, the famous paradox of the British mathematician-philosopher B. Russell exposed the contradictions of the set theory itself, which shook the foundation of mathematics and caused the third crisis in the history of the development of mathematics.

To avoid the paradox and solve the problem of the foundation of set theory itself, the German mathematician E. Zermelo proposed an axiomatic system, which was later improved by A. Fraenkel and others to form the currently recognized Zermelo Fraenkel axiomatic system, or ZF axiomatic system for short. Zermelo's proposal of axiomatic system of set theory included the axiom of choice which is also an issue that has caused a long debate in the mathematical community. The axiom of choice, like the continuum hypothesis, is a problem caused by infinity. There are more than 20 equivalent forms of the axiom of choice, such as the well-ordering theorem and the intersection uniqueness theorem. The simpler one is given here. Axiom of Choice (Intersection Uniqueness Theorem): let Ω be a nonempty set consisting of nonempty sets that do not intersect each other, then there exists a set A, such that $\forall S \in \Omega, A \cap S$ would be singleton.

In 1878, writing in the Journal of Mathematics, Cantor posed the continuum problem of whether there are infinite subsets of R with potentials other than the potential of the set of natural numbers N and the potential of the set of real numbers R.

Cantor argued that the conclusion was negative, i.e., that any infinite subset of R is either countable or has the potential of R. This is Cantor's Continuum Hypothesis, often notated as CH (Continuum Hypothesis, may be referred to as the hypothesis in the following paragraph).

Hilbert's interest in and influence on the hypothesis was nowhere greater than when he presented it as the 1st problem at the International Mathematical Congress in Paris in 1900. Prior to that, he had discussed the problem directly with Cantor on several occasions, having fully recognized its significance and difficulty. On June 4, 1925, Hilbert delivered his famous lecture in Münster entitled

"On Infinity", in which he attempted to solve the continuum problem using proof-theoretic principles. However, his proof was framed, poorly argued, and incorrectly asserted, and was soon criticized. Gödel's entry point for exploring the problem was Hilbert's proof outline. He learned of the outline in the summer of 1930, just as he was discovering the First Incompleteness Theorem. Hilbert's method is to construct (recursive) ordinals and then apply sets and functions with recursive definitions that are ordered by such ordinals. The proof used the erroneous proposition that every subset of the set of natural numbers is a recursive set. Gödel, inspired by Hilbert's line of proof, also used the method of ordering (integer) sets by ordinal numbers, but he avoided recursively defined sets and instead used first-order definable sets from the branching spectrum. It was in 1935 that Gödel realized that formable sets satisfy all the axioms of set theory, including the axiom of choice. That year he proved the relative compatibility of the axiom of choice, and later he proved the relative compatibility of the generalized continuum hypothesis by introducing the axiom of compatibility.

In 1963, Cohen of Stanford University proved that if the ZF system is compatible, both the axiom of choice and the continuum hypothesis are independent of this axiomatic system. In other words, the axiom of choice and the continuum hypothesis cannot be introduced from the ZF system. Moreover, he proved that even if the axiom of choice selection is added to the ZF system, the continuum hypothesis, and thus the generalized continuum hypothesis, still cannot be proved. Combined with the relative compatibility established by Gödel in 1938, this proves that CH is undecidable in the sense of Entscheidungs problem in the ZF system [7].

P. J. Cohen introduced a new meta-mathematical concept of the Forcing method. He started by making a very small ZF model, and then added some elements to make it satisfy the set axioms, but contradict other specific statements. Among other things, in particular, he proved that the ZF axiom and the axiom of choice can be satisfied simultaneously and that $2^{\aleph_0}$ could be any value. For example, ℵ_2, ℵ_3, ℵ_(ω+1) are all possible, so there are many ways of violating CH. Cohen cleverly borrowed from Skowron's paradox that any formal system can be interpreted in a countable model, and the models he gave have the properties of non-standard models. He actually proves that every assumption of a power of the regular base is compatible with the ZF system. This shows us the possibilities of making the intuitive notion of a set precise. The incompleteness of set-theoretic axioms to such an extent is unexpected, and implies that there are infinitely many ZF set-theoretic systems in which the potential of the continuum is not the same in any two of them; and that if the original ZF set theory is compatible, then all of these systems are also compatible

The results of Gödel and Cohen show that CH is independent of ZFC. This means that it is consistent to have $\mathrm{ZFC} + \mathrm{CH}$ and it is also consistent to have $\mathrm{ZFC} + \neg\mathrm{CH}$. The proof of these consistency facts proceeds by building models. one can construct one model $\mathrm{M} \vDash \mathrm{ZFC} + \mathrm{CH}$ and another model $\mathrm{N} \vDash \mathrm{ZFC} + \neg\mathrm{CH}$. In fact, there are models in which the continuum $2^{\aleph_0}$ is anything not contradicted by Cantor's theorem that $2^{\aleph_0} > \aleph_0$ or by the somewhat later result of König that $2^{\aleph_0} \neq \aleph_\gamma$ if γ is a limit ordinal of countable cofinality, such as $\omega, \omega^3 + \omega^2, \omega_1 + \omega$, etc. But, $2^{\aleph_0}$ can be $\aleph_2$ or $\aleph_{17}$ or $\aleph_{\omega+1}$ or $\aleph_{\omega_1}$ [8].

3.2. Summary and overview of recent research progress

The discussion on CH continues to this day. Mathematicians and logicians can be grouped into four main perspectives based on their views on how CH should be addressed: Formalism, Platonism, Multiverse View, and Universe-Indeterminism.

3.2.1. Formalism and Platonism. Formalists believe that mathematics is a formal computation based on axioms. From this perspective, if CH cannot be concluded from the current axiomatic system, then it is neither true nor false but indeterminate [9]. Paul Joseph Cohen is one of the notable representatives of Formalism, who believes that scholars should be content with the current status of CH, considering CH as already resolved. In stark contrast, Kurt Gödel, a famous Platonist, holds the view that CH is a meaningful question about the set-theoretic universe, and it must have a truth value. The current situation of CH merely suggests that our knowledge of the set-theoretic universe is very limited. Gödel's perspective can be summarized by the general principle known as Gödel's Program, which seeks to solve mathematically meaningful questions independent of ZFC by appropriately strengthening the axioms of ZFC. Another Platonist, Hugh Woodin, aligns with Gödel's Program and proposes the Woodin's Hypothesis, which states that if the set of all stationary sets has a base of cardinality $\aleph^1$, then $2^{\aleph_0} = \aleph_2$. The following is the theoretical foundation of the Woodin's Hypothesis:

It is claimed that B is a basis for infinite linear orders if and only if every infinite linear order$(L, <)$ has a substructure L $(L', <)$ that is isomorphic to an element in T. All infinite linear orders start with 2, thus it's clear that this is the case. In a recent proof, Justin Moore established that there exists a set of uncountable linear orders of cardinality 5. Moore deduced from this that $2^{\aleph_0} = \aleph_2$ holds if and only if all uncountable linear orders have a basis with cardinality smaller than $2^{\aleph_1}$.

Then, take all stationary sets into account. So long as there exists a set X such that for each stationary set S, X exists so that for any unbounded closed set C, $C \cap T \subset C \cap S$., then X is said to be a base for all stationary sets. It is compatible with the Large Cardinal Axiom to have a base for all stationary sets of cardinalities if and only if the Large Cardinal Axiom is consistent. Saharon Shelah showed that $2^{\aleph_0} = \aleph_1$ if and only if all stationary sets have cardinality $\aleph_1$. Woodin formulated Woodin's Hypothesis on this basis.

Chris Freiling, another representative of Platonism, proposed in 1986 an argument based on the Axiom of Symmetry: $\forall f: R \to R_{\aleph_0},\ f(x) = S_x \subset R,\ \exists x, y \in R$ such that $y \notin f(x)$ and $x \notin f(y)$ [10]. This argument primarily uses probabilistic intuition to oppose CH. To illustrate his point, Freiling employed the following thought experiment:

Take into account everything on the actual number line. Pick two more (potentially overlapping) spots at random and evenly from each vantage point. Now, think about the location x. Since CH claims that the cardinality of the reals is the same as twice the cardinality of the naturals, leading to this oddity, the probability that the two chosen points fall within any given open interval around x is zero. However, since this is true for every point, we would expect that, for almost every point x, the likelihood that its chosen points fall in the interval of another point is extremely low. Intuition, however, informs us that virtually all points will have at least one of the points it selects lie inside the interval another points selects. This goes against what CH predicts.

Freiling's Symmetry Axiom captures the above intuition and formalizes it as: For any function f mapping real numbers to ordered pairs of real numbers, for almost all x, there exists a y such that x is not in $f(y)$ and y is not in $f(x)$ [10]. The idea behind this intuition is that if CH were true, then the aforementioned symmetry would not occur. However, our intuition suggests that this symmetry should exist, leading Freiling to the conclusion that CH should be false. Nonetheless, it's important to note that this argument heavily relies on probabilistic intuition, and not all mathematicians agree with this perspective.

3.2.2. Multiverse View. The set-theoretic multiverse perspective is an emerging philosophical stance in set theory. It opposes realism, asserting that there's no absolute set-theoretic universe and no canonical concept of set, hence independent propositions like CH don't have a so-called correct answer (Antos, 2015). Currently, various set theorists have preferences for working within certain models or types of models. For instance, Ronald Jensen prefers to operate under the assumption of $V = L$; proponents of the Gödel program are often accustomed to proving theorems under the assumptions of large cardinals, and so on. However, the work of all these set theorists is mutually acknowledged because they can all be interpreted, at least, as proofs in the ZF or ZFC systems concerning some hypothetical propositions. For instance, if $ZFC + V = L$ can prove φ, then ZFC can prove "$V = L$ implies φ" [11].

This phenomenon is viewed by multiverse proponents as one of the best evidence supporting the set-theoretic multiverse perspective. Scholars have experience "living" within various set-theoretic universes. These experiences are clear and reliable enough that it's challenging to easily dictate which set-theoretic model reflects the so-called "true" set-theoretic universe. The inventor of the "multiverse" perspective, Joel David Hamkins notes that, given these realities of set-theoretic development, an ideal resolution to CH for realists has become unattainable. Instead, contemporary set-theoretic research should focus on understanding how independent propositions hold in set-theoretic universes, as well as the relationships between these universes. Several research findings or directions are considered to be inspired by the multiverse perspective. Two typical new research directions that have emerged are the modal logic of forcing and set-theoretic geology. In conclusion, Hamkins adopts a defense strategy based on its practical utility, that is, the existence of the multiverse has practical value, and thus should be acknowledged [11].

3.2.3. Universe-Indeterminism. The view that standard set theories deploy inherently indeterminate concepts has been championed by Solomon Feferman. Standard set theories posit a singular universe of sets, known as the cumulative hierarchy [11]. Yet, the iterative model and its formal representations are based on concepts with inherent ambiguity. This uncertainty originates from the notion that, at each subsequent step, scholars aggregate all conceivable sets from prior stages; this notion stems from endorsing the powerset axiom and its incorporation into the cumulative hierarchy's recursive definition. Owing to this inherent vagueness, the complete universe of sets fails to ascribe definite truth values to propositions in set theory, including the likes of CH and GCH [11].

4. Conclusion

It is hard to commit to the possibility that ZFC provides all the insights, even if one suggests that the transition from one infinite cardinal to another involves mechanisms more intricate than just the powerset. For the reason that that possibility is challenged by multiverse and other theories. Infinity is not a constituent of any of the numerical systems that are often associated with the concept of numbers. This assertion is true for the sets of natural numbers, integers, rational numbers, real numbers, and ordinal numbers. The encapsulation of the notion of infinity may be achieved by the amalgamation of cardinal and ordinal numbers, or through mathematical operations performed on these numbers. However, a thorough knowledge of mathematicians' perspectives on infinity cannot be achieved only by focusing on topics like cardinals or ordinals. The incorporation of set theory is a crucial aspect to be considered in the context of this matter.

References

[1] Woodin WH. The realm of the infinite. Infinity. New research frontiers. 2011:89-118.
[2] Antos C., Friedman, S.-D., Honzik, R., and Ternullo, C. 2015. Multiverse conceptions in set theory. Syn. (Dordrecht), 192(8), 2463–2488. https://doi.org/10.1007/s11229-015-0819-9
[3] Hilbert D. 1926 On the infinite. Mat. Anl. 95 161-90.
[4] Li Na and Ye Fayan. 2022 Hamkins' set-theoretic multiverse view and its defense strategy. Studies in Natural Dialectics 12 97-102. doi:10.19484/j.cnki.1000-8934.2022.12.003.
[5] Hao Zhaokuan, Shi Xianghui & Yang Yue. 2010 The continuum problem and the Ω-conjecture. Studies in Logic 04 30-43.
[6] Meo M. 2004 The mathematical life of Cauchy's group theorem. Historia Mathematica. 31 196-221.
[7] Kunen K. 1980 Set theory an introduction to independence proofs. North-Holland Pub. Co.
[8] Hao, Zhaoguan & Yang, Yue. 2017 Platonism and the set-theoretic ultimate universe. Studies in Logic 04 87-98
[9] Morton AB. 2013 Throwing Darts at a Ruler: Unpacking the Intuition Behind Freiling's Axiom of Symmetry. arXiv.org.
[10] Yang, Rui Zhi. 2015 A review of set-theoretic multiverse view. Studies in Natural Dialectics 09 99-103.
[11] Scambler C. An indeterminate universe of sets. Syn. (Dordrecht), 2020 545.