

Group theory behind Rubik's Cube

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Abstract. The Rubik's Cube is a widely recognized puzzle. The mathematics behind the Rubik's Cube is group theory. Group theory studies algebraic structures in mathematics such as groups, rings, and fields. The operation of the Rubik's Cube is rotation, which can be considered an operation of a group. The combination of two rotations of the Rubik's Cube can be considered the association of two operations of a group. The rotations and the combination operation of two rotations form a group called the Rubik's Cube group, and this paper presents the order of this group which is also the quantity of possible valid configurations of the Rubik's Cube. The valid configurations are the configurations that can be reached by a series of rotations from the starting configuration. This paper presents a method to illustrate the configurations of the Rubik's Cube, the requirements for making the configurations valid, and calculate the quantity of possible valid configurations.

Keywords: Group theory, Rubik's Cube, Sign Homomorphism.

1. Introduction

The Rubik's Cube is a three-dimensional puzzle initially created by architect Ernő Rubik, and it is one of the best-selling toys in the world. The first mathematical approach to analyzing Rubik's Cube was written by mathematician David Singmaster [1]. The Rubik's Cube is still widely recognized today. The competition that requires players to solve Rubik's Cube as quickly as possible is called speed cubing, and such competitions are held worldwide every year today. The world record up to 2023 for solving a single Rubik's Cube is 3.13 seconds [2].

The mathematics that Singmaster first uses to find a general step-by-step solution to Rubik's Cube is group theory, which research on algebraic structures. A group is a structure that consists of a nonempty set and a binary operation. In Rubik's Cube, all rotations and the combination of two rotations form a group, and this group is called the Rubik's Cube group. Every rotation can be thought of as an element of the Rubik's Cube group, the move that does not change the current configuration can be considered as the identity element of this group, and the combination of two rotations can be thought of as the association of two elements in this group. The solution to the Rubik's Cube can be illustrated clearly via group theory and group theory can be used to examine whether an arbitrary configuration of the Rubik's Cube can be returned to the start configuration through some rotations of the cube (is valid) or not. To study the restrictions that make a configuration valid, the notion of group homomorphism and permutation is required [3]. Group homomorphism studies the relationship between two groups and permutation is a bijection of a nonempty set. It can be easily understood that the order of the Rubik's

Cube equals the quantity of the possible valid configurations, and this quantity can be calculated with the restrictions enabling a configuration valid.

In the twenty-first century, the problem of finding the minimum distance between two cycles of the cube is a problem requiring not only mathematics but also computer science. God's number is the highest value of the optimal solution of Rubik's Cube. God's number was continually updated in the first decade of the twenty-first century by researchers using algorithms and the increasing use of computational resources. God's number was eventually determined and proven to be 20 in 2010 using computers at Google and required a total of 35 CPU years [4].

The mathematics requires to illustrate the solutions to the Rubik's cube are basic concepts and theorems. Specifically, the notions of group, subgroup, group homomorphism, and permutation [5]. In the calculation process of the possible configurations, the knowledge of basic combinatorics is also required. In this paper, the solution to Rubik's Cube via group theory is presented, and the number of possible valid configurations is calculated.

2. Preliminary

2.1. Groups

Rubik's Cube contains a mathematical structure called a group. The concept of the group is a mathematical structure that satisfies three properties.

Definition 1. Suppose G is a nonempty set. A binary operation $*$ on G is a function such that:

$$*: G \times G \rightarrow G, (g_1, g_2) \mapsto g_1 * g_2. \quad (1)$$

$g_1 * g_2$ is usually shortened to $g_1 g_2$.

Definition 2. A group is a nonempty set G together with a binary operation $*$ on it (denoted by G or $(G, *)$) such that:

The binary operation $*$ is associative: $\forall g_1, g_2, g_3 \in G, (g_1 g_2) g_3 = g_1 (g_2 g_3)$. G has an identity element id_G : $\forall g_1 \in G, \exists id \in G$ such that $g_1 id_G = id_G g_1 = g_1$. Existing inverse element: For any $g_1 \in G, \exists g_1^{-1} \in G$ such that $g_1^{-1} g_1 = g_1 g_1^{-1} = id_G$.

Definition 3. Suppose G is a group. A subgroup H of G (denoted as $H < G$) is a subset of G such that: H is closed: $\forall g_1, g_2 \in H, g_1 g_2 \in G$. Existing inverse element: $\forall g_1 \in G, \exists g_1^{-1} \in G$ such that $g_1^{-1} g_1 = g_1 g_1^{-1} = id_G$.

Definition 4. Suppose G is a group, the order of G is the number of elements in G , and denoted by $|G|$.

Definition 5. Suppose G is a group, and $H < G$. $\forall g_1 \in G$, define $g_1 H := \{g_1 h : h \in H\}$ and $H g_1 := \{h a : h \in H\}$. Then $g_1 H$ is a left coset of H , $H g_1$ is a right coset of H , and both $g_1 H$ and $H g_1$ are called cosets of H .

Definition 6. Suppose n is a positive integer. For any $x \in \{0, 1, \dots, n - 1\}$, let \bar{x} denotes the coset of $n\mathbb{Z}$ such that:

$$\bar{x} = x + n\mathbb{Z}. \quad (2)$$

Define the addition between the cosets of $n\mathbb{Z}$ as $\bar{x} + \bar{y} = \begin{cases} \overline{x + y}, & x + y < n; \\ \overline{x + y - n}, & x + y \geq n. \end{cases}$

$\mathbb{Z}/n\mathbb{Z}$ forms a group with cosets $\{\overline{0}, \overline{1}, \dots, \overline{n - 1}\}$ and the addition defined above, and $\mathbb{Z}/n\mathbb{Z}$ is called the group of addition modulo n [6].

2.2. Permutation

The permutation group contributes to illustrating the structure of the Rubik's Cube in a clear way with positive integers.

Definition 7. Let G be a nonempty set. The set of all bijections $G \rightarrow G$ is called the symmetric group on G , and denoted by $Sym(G)$. When $G = \{1, 2, 3, \dots, n\}$, $Sym(G)$ is written as S_n , and S_n is called symmetric group on n letters.

Definition 8. A permutation of a nonempty set G is a bijection from G to G . Suppose $\sigma \in S_n$ is a permutation, $\forall i \in \{1, 2, 3, \dots, n\}, i \mapsto \sigma(i)$.

Definition 9. The cycle $(i_1 i_2 \dots i_k)$ is an element σ of S_n defined by $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \dots, \sigma(i_{k-1}) = i_k, \sigma(i_k) = i_1$ and $\sigma(j) = j$ for $j \in \{1, 2, 3, \dots, n\} \setminus \{i_1, i_2, \dots, i_k\}$. The length of $(i_1 i_2 \dots i_k)$ is k . Let n be a positive integer, a cycle of length n is called a n -cycle.

Definition 10. Suppose σ and τ are two cycles of S_n . σ and τ are disjoint if $\text{supp}\sigma \cap \text{supp}\tau = \emptyset$.

Theorem 1. Every permutation $\sigma \in S_n$ can be expressed as a unique product of a finite number of pairwise disjoint cycles [7].

Proof. Suppose $i \in \{1, 2, 3, \dots, n\}$. Then consider the sequence $i, \sigma(i), \sigma^2(i), \dots$. This sequence must eventually repeat since for every $j \in \{1, 2, 3, \dots, n\}, \sigma(j) \in \{1, 2, 3, \dots, n\}$. Therefore, $\exists q > p$ such that $\sigma^p(i) = \sigma^q(i)$, meaning $\sigma^{q-p}(i) = i$. Let $k = q - p$. Then $(i \sigma(i) \sigma^2(i) \dots \sigma^{k-1}(i))$ is a cycle in S_n , which means every element of the set $\{1, 2, 3, \dots, n\}$ belongs to a cycle in S_n . Moreover, any cycle contains i must be the same cycle shown above.

Definition 11. Even permutations are the permutations in S_n that can be presented as a product of some 2-cycles. Odd permutations are the permutations in S_n that are not even permutations.

2.3. Homomorphism

The homomorphism illustrates the relationship between two groups, the sign homomorphism below is of great importance in determining whether a configuration can be returned to the starting configuration after some rotations or not.

Definition 12. Let $(G, *)$, and (H, \cdot) be two groups. A homomorphism from G to H is a map $f: G \rightarrow H$, $\forall p, q \in G, f(p * q) = f(p) \cdot f(q)$. f is called an isomorphism if f is bijective.

Theorem 2. Let a_1, a_2, \dots, a_n be positive integers. Define $\Delta(a_1, a_2, \dots, a_n) = \prod_{1 \leq i < j \leq n} (a_i - a_j)$. Suppose $\sigma \in S_n$, define $\sigma(\Delta(a_1, a_2, \dots, a_n)) = \Delta(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)})$. With such definition, $\sigma(\Delta(a_1, a_2, \dots, a_n)) = \pm \Delta(a_1, a_2, \dots, a_n)$ [8].

Proof. By the definitions given above,

$$\sigma(\Delta(a_1, a_2, \dots, a_n)) = \Delta(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}) = \prod_{1 \leq i < j \leq n} (a_{\sigma(i)} - a_{\sigma(j)}) \quad (3)$$

$$\Delta(a_1, a_2, \dots, a_n) = \prod_{1 \leq i < j \leq n} (a_i - a_j) \quad (4)$$

Since σ is a bijection, for any $1 \leq i < j \leq n, \sigma(i) \neq \sigma(j)$. Let i' be the smaller number between $\sigma(i)$ and $\sigma(j)$, and j' be the other number, then $a_{\sigma(i)} - a_{\sigma(j)} = \pm(a_{i'} - a_{j'})$. Therefore, the theorem holds since $\sigma(\Delta(a_1, a_2, \dots, a_n)) = \Delta(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}) = \prod_{1 \leq i < j \leq n} (a_{\sigma(i)} - a_{\sigma(j)}) = \prod_{1 \leq i < j \leq n} \pm (a_i - a_j) = \pm \prod_{1 \leq i < j \leq n} (a_i - a_j) = \pm \Delta(a_1, a_2, \dots, a_n)$.

Lemma 1. Let a_1, a_2, \dots, a_n be positive integers, suppose $\sigma, \tau \in S_n$. Then

$$(\sigma\tau)(\Delta(a_1, a_2, \dots, a_n)) = \sigma(\tau(\Delta(a_1, a_2, \dots, a_n))) \quad (5)$$

Proof. Since $(\sigma\tau)(\Delta(a_1, a_2, \dots, a_n)) = \Delta(a_{\sigma\tau(1)}, a_{\sigma\tau(2)}, \dots, a_{\sigma\tau(n)})$ and $\forall 1 \leq i \leq n, \sigma\tau(i) = \sigma(\tau(i))$; then $(\sigma\tau)(\Delta(a_1, a_2, \dots, a_n)) = \sigma(\Delta(a_{\tau(1)}, a_{\tau(2)}, \dots, a_{\tau(n)})) = \sigma(\tau(\Delta(a_1, a_2, \dots, a_n)))$.

Definition 13. Suppose $\sigma \in S_n, a_1, a_2, \dots, a_n$ be positive integers. Define

$$\text{sgn}: S_n \rightarrow \{1, -1\}, \sigma(\Delta(a_1, a_2, \dots, a_n)) = \text{sgn}(\sigma)\Delta(a_1, a_2, \dots, a_n). \quad (6)$$

Theorem 3. sgn is a homomorphism and is called Sign Homomorphism.

Proof. Suppose $\sigma, \tau \in S_n, a_1, a_2, \dots, a_n$ be positive integers. From **Lemma 1**,

$$\begin{aligned} (\sigma\tau)(\Delta(a_1, a_2, \dots, a_n)) &= \sigma(\tau(\Delta(a_1, a_2, \dots, a_n))) = \text{sgn}(\sigma\tau)\Delta(a_1, a_2, \dots, a_n) \\ &= \text{sgn}(\sigma)\Delta(a_{\tau(1)}, a_{\tau(2)}, \dots, a_{\tau(n)}) = \text{sgn}(\sigma)\text{sgn}(\tau)\Delta(a_1, a_2, \dots, a_n) \end{aligned} \quad (7)$$

Therefore, $\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$, meaning sgn is a homomorphism.

3. Rubik's Cube

3.1. Notations of the Rubik's Cube

In all the following contents below, the cube refers to the Rubik's Cube and the configurations and faces refer to the configurations and faces of the Rubik's Cube.

Definition 14. The 27 small cubes that form the cube are called cubies. Corner cubies are the cubies located at the corner of the cube, and edge cubies are the cubies located at the edge of the cube.

Definition 15. Cubicles are the locations that cubies stay in the cube.

To illustrate the rotations of the cube clearly, some notations are needed. The six faces of the cube, which are the upward, downward, right, left, front and back face, are denoted as $u, d, r, l, f,$ and b respectively. These notations are introduced by David Singmaster [1].

Example 1. The cubie in the front, upper, and left corner is denoted as ful .

Definition 16. The rotations of the cube are called moves.

Fixing the cube, then the moves that rotate every face of the Rubik's Cube 90 degrees clockwise are called fundamental moves and denoted as follows. Looking at the upward face then rotate the front face 90 degrees clockwise. This rotation is denoted as U . The rest of the moves of the downward, right, left, front, and back faces are similarly denoted as D, R, L, F, B respectively.

Definition 17. The operation that performs one move of the cube first and then performs another move of the cube is called concatenation.

3.2. The Possible Configurations

The cube has 12 edge cubies and 8 corner cubies. It can be observed that any move of the cube does not change any center cubie since the orientation and location of the center cubie that is affected by the move remain the same. Additionally, after moving the Rubik's Cube, corner cubicles are the only locations that corner cubies can be in, and edge cubicles are only locations that edge cubies can be in. With these observations, each of the 8 corner cubies has 3 orientations, and the 8 corner cubies have $8!$ arrangements to be put into the 8 corner cubicles. Therefore, there exists $3^8 \cdot 8!$ possible configurations of corner cubies. Similarly, each of the 12 edge cubies has 2 orientations, and the 12 cubies have $12!$ arrangements to be put into the 12 cubicles, meaning the edge cubies have $2^{12} \cdot 12!$ possible configurations. Therefore, the cube has in total of $3^8 \cdot 2^{12} \cdot 8! \cdot 12!$ possible configurations [9].

Definition 18. A configuration is valid if it can be obtained from the starting configuration after a series of fundamental moves.

3.3. The Rubik's Cube Group

Theorem 4. The set of all moves that can be obtained from the starting configuration by performing the rotations of the cube with the binary operation concatenation forms a group. This group is called the Rubik's Cube group and is denoted by (\mathbb{C}, \cdot) [10].

Proof. Concatenation is associative since performing two moves and then performing another move equals performing one move first and then performing the other two move.

Suppose \mathcal{R} is an arbitrary move.

The move that does not change the configuration of the cube is the identity of (\mathbb{C}, \cdot) and is denoted by $id_{\mathbb{C}}$ since $\mathcal{R} \cdot id_{\mathbb{C}} = id_{\mathbb{C}} \cdot \mathcal{R} = \mathcal{R}$. The move that reverses of all the rotations of \mathcal{R} is the inverse element of \mathcal{R} and is denoted by \mathcal{R}^{-1} since $\mathcal{R} \cdot \mathcal{R}^{-1} = \mathcal{R}^{-1} \cdot \mathcal{R} = id_{\mathbb{C}}$.

3.4. The Valid Configurations

From 3.2., a configuration is determined by the orientations and locations of the corner and edge cubies. Since the eight corner cubicles are the only places that the eight corner cubies can be in, the locations of the corner cubies can be described by S_8 . Similarly, the locations of the edge cubies can be described by S_{12} . For each corner cubie, there exist three orientations that can be distributed to three faces of the corner cubie.

For each corner cubie, write 0,1,2 on the three faces of the cubie, more specifically, suppose every corner cubie is in the *dlf* cubicle, then write 0 to the d face, 1 to the l face, 2 to the f face. Mark the upward and downward faces of all the corner cubicles with '+'. The numbers on the corner cubie face that are marked with '+' are assigned to each corner cubie. Therefore, the orientations of the corner cubies of a configuration can be described as an 8 tuple of $(\mathbb{Z}/3\mathbb{Z})^8$. For each edge cubie, write 0,1 on the two faces of the cubie, more specifically, suppose every edge cubie is in the *lf* cubicle, then write 0 to the l face, 1 to the f face. Mark the upward, downward, front, and back faces of all the edge cubicles with ' χ '. The numbers on the edge cubie face that are marked with ' χ ' are assigned to each edge cubie. Therefore, the orientations of the edge cubies of a configuration can be described as a 12 tuple of $(\mathbb{Z}/2\mathbb{Z})^{12}$.

Theorem 5. With the above discussion, it is clear that a configuration can be specified by four data: The locations of corner cubies $\sigma \in S_8$, the locations of the edge cubies $\tau \in S_{12}$, the orientations of the corner cubies $x \in (\mathbb{Z}/3\mathbb{Z})^8$, and the orientations of the edge cubies $y \in (\mathbb{Z}/2\mathbb{Z})^{12}$ ($x \in (\mathbb{Z}/3\mathbb{Z})^8$, $x = (x_1, x_2, \dots, x_8)$ ($1 \leq i \leq 8, \forall x_i \in \mathbb{Z}/3\mathbb{Z}$). $y \in (\mathbb{Z}/2\mathbb{Z})^{12}$, $y = (y_1, y_2, \dots, y_{12})$ ($1 \leq i \leq 12, \forall y_i \in \mathbb{Z}/2\mathbb{Z}$). Therefore, a configuration can be presented as a tuple: $(\sigma, \tau, x, y) \in S_8 \times S_{12} \times (\mathbb{Z}/3\mathbb{Z})^8 \times (\mathbb{Z}/2\mathbb{Z})^{12}$.

Theorem 6. Let $(\sigma, \tau, x, y) \in S_8 \times S_{12} \times (\mathbb{Z}/3\mathbb{Z})^8 \times (\mathbb{Z}/2\mathbb{Z})^{12}$. (σ, τ, x, y) is a valid configuration if and only if: $sgn(\sigma) = sgn(\tau)$; $\sum_{i=1}^8 (x_i) \equiv 0 \pmod{3}$; $\sum_{i=1}^{12} (y_i) \equiv 0 \pmod{2}$.

Proof. Firstly, suppose (σ, τ, x, y) is a valid configuration. Let $M \in (\mathbb{C}, \cdot)$ be the move that make the Rubik's Cube from the starting configuration to (σ, τ, x, y) , $M = M_1 \cdot M_2 \cdot \dots \cdot M_k$ ($\forall 1 \leq i \leq k, M_i$ is a fundamental move).

Since any fundamental move changes four corner and edge cubies, and such move is a 4-cycle which has $sgn = -1$, and for any fundamental move \mathcal{M} , $sgn(\mathcal{M}) = -1$. Therefore, $sgn(\sigma) = sgn(\prod_{i=1}^k M_k) = \prod_{i=1}^k sgn(M_i) = sgn(\tau) = -1$.

To prove that $\sum_{i=1}^8 (x_i) \equiv 0 \pmod{3}$ and $\sum_{i=1}^{12} (y_i) \equiv 0 \pmod{2}$ holds, the data of the changes after performing fundamental moves is needed since M is the concatenation of some of these moves.

Table 1. The changes to x and y after each fundamental move.

fundamental move	The changes to x and y after each fundamental move
U	$(x_2, x_3, x_4, x_1, x_5, x_6, x_7, x_8)$ $(y_4, y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$
D	$(x_1, x_2, x_3, x_4, x_8, x_5, x_6, x_7)$ $(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_{10}, y_{11}, y_{12}, y_9)$
R	$(x_1, x_7 + 1, x_2 + 2, x_4, x_5, x_6, x_8 + 2, x_3 + 1)$ $(y_1, y_7, y_3, y_4, y_5, y_2, y_{10}, y_8, y_9, y_6, y_{11}, y_{12})$
L	$(x_4 + 2, x_2, x_3, x_5 + 1, x_6 + 2, x_1 + 1, x_7, x_8)$ $(y_1, y_2, y_3, y_5, y_{12}, y_6, y_7, y_4, y_9, y_{10}, y_{11}, y_8)$
F	$(x_6 + 1, x_1 + 2, x_3, x_4, x_5, x_7 + 2, x_2 + 1, x_8)$ $(y_1, y_2, y_8 + 1, y_4, y_5, y_6, y_3 + 1, y_{11} + 1, y_9, y_{10}, y_7 + 1, y_{12})$
B	$(x_1, x_2, x_8 + 1, x_3 + 2, x_4 + 1, x_6, x_7, x_5 + 2)$ $(y_6 + 1, y_2, y_3, y_4, y_1 + 1, y_9 + 1, y_7, y_8, y_5 + 1, y_{10}, y_{11}, y_{12})$

From **Table 1**, it can be calculated that every fundamental move satisfies $\sum_{i=1}^8 (x'_i) \equiv 0 \pmod{3}$ and $\sum_{i=1}^{12} (y'_i) \equiv 0 \pmod{2}$. Since $M = M_1 \cdot M_2 \cdot \dots \cdot M_k$, $\forall 1 \leq i \leq k, M_i \in \{F, B, R, L, U, D\}$, then $\sum_{i=1}^8 (x_i) \equiv 0 \pmod{3}$ and $\sum_{i=1}^{12} (y_i) \equiv 0 \pmod{2}$.

One direction of the proof has been shown. Showing the other direction of the proof is more difficult, a complete proof can be found in [5]. The general steps of proving the other direction are the following:

Put all the corner cubies in the correct cubicles. Put all the corner cubicles in the correct cubicles with the correct orientations. Put all the edge cubies in the correct cubicles and remain the corner cubies unchanged. Put all the edge cubies in the correct cubicles with the correct orientations and remain the corner cubies unchanged. Putting a cubie in the correct cubicle means the faces of the cubicle are the same as the faces of the cubie.

Theorem 7. The Rubik's Cube group has the order $\frac{3^8 \cdot 2^{12} \cdot 8! \cdot 12!}{12}$.

Proof. From 3.2, the quantity of possible configurations is:

$$3^8 \cdot 2^{12} \cdot 8! \cdot 12! \quad (8)$$

From **Theorem 6.** only $\frac{1}{12}$ of the possible configurations are valid since:

There are the same number of odd permutations as the even permutations. The *sgn* of all odd permutations is -1 and the *sgn* of all even permutations is 1. Therefore, the possible configurations should be deduced $\frac{1}{2}$. Since the orientation of the 8th corner cubie is determined by the orientations of the rest of 7 corner cubies, the possible configurations should be deduced $\frac{1}{3}$. Since the orientation of the 12th edge cubie is determined by the orientations of the rest of 11 edge cubies, the possible configurations should be deduced $\frac{1}{2}$.

The order of the Rubik's Cube group is also the quantity of the possible valid configurations of the cube.

4. Conclusion

To mathematically solve the Rubik's Cube, basic notions of group theory are needed. The rotations of the cube can be considered as permutations, and to specify the configurations, two sets of numbers are assigned to the corner cubies and edge cubies respectively. The rotations together with the binary operation concatenation form a group. The binary operation concatenation of two rotations is doing one rotation first, and then doing the other. A configuration is determined by the corner and edge cubies, specifically by their orientations and locations. With the Rubik's Cube and the assigned numbers on the corner and edge cubies, the requirements of the valid configurations can be proved. With the requirements of the valid configurations, the possible valid configurations can be calculated.

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