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A method to test the uniform convergence of function series

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Abstract. The series refer to performing infinite addition operations on infinite numbers or functions in a certain order. It is hard to find out whether the positive function series converges uniformly in many cases. In this article, a new method that replacing the sum of function terms series with improper integral will be introduced, which is designed to solve problems that cannot be solved by classical Weierstrass M-test. The Cauchy uniform convergence test will serve as the basis for the entire proof process because it can lead the focus point from the whole sum to the partial sum of the function series, where its value can be easier substituting by the value of the improper integral. After using basic knowledge of the improper integral, the uniform convergence can finally be known. By using this method, testing the uniform convergence of the irregular function series even estimating its value can be possible accomplished.

Keywords: Uniform convergence of function series, Improper integral, Cauchy's convergence test, Weierstrass M-test, Mathematical analysis.

1. Introduction

A series is a sequence of countable real numbers and it is important to study its sum [1]. According to historical records, Archimedes was the first person to give the sum of an infinite series. When calculating the area under the arc of the parabola, he used the exhaustive method, which extremely approximated the value of π [2-3]. However, people later realized that testing the convergence of the series rather than directly calculating the sum of the series could indirectly understand the properties of a series. After that, the sages have devoted themselves to the study of series convergence.

The function series as the topic of this paper is a series, where the terms are functions. Among the various types of convergence, the uniform convergence is very ideal for a series because many properties of the function series are preserved by its convergent function [4]. If a function series is equicontinuous, then the property of continuity transfers to the limit function. Cauchy firstly came out with the theory of uniform convergence. Later, Seidel and Stokes pointed out Cauchy's limitations [5]. Cauchy then acknowledged their advice and reached the Stokes' conclusions [6]. Thomae used Cauchy's theory for his own theory of functions without realizing in time the difference between uniform convergence and non-uniform convergence [7]. The Weierstrass M-test is also helpful to test the uniform convergence of function series, but this is not a universal method [8]. Florentin used improper integral to approximate the value of positive series, but the method of using improper integral to determine the series of function terms has not yet appeared [9]. The subject of the paper is to give a method of testing about function term series.

The paper is organized as following. In section 2, the basic knowledge will be shown. In section 3, the proof of this method will be given. In section 4, two applications by using this method will be displayed.

2. Basic Knowledge

Let's recall some facts of improper integral and function series that will be applied for the proof.

Theorem 2.1 (Cauchy's convergence test) For every ε > 0, if $\sum_{k=1}^{\infty} u_k$(x) is uniformly convergent, then there exists a natural number N and for every p∈N, when n > N,

$$u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon \tag{1}$$

Example 2.2: Let u_k(x) =x^k, for all x∈ [0, ρ], 0 < ρ< 1. Prove that the series is uniform convergent.

Proof:

Since u_k(x) $\le \rho^k$. For all k∈N, there exists $| u_{n+1} + u_{n+2} + \cdots + u_{n+p} | \le \rho^{n+1} + \rho^{n+2} + \ldots + \rho^{n+p} = \rho^{n+1}(1 + \rho + \ldots + \rho^{p-1}) = \rho^{n+1}\frac{1-\rho^p}{1-\rho}$. Since 0 < ρ < 1, as n →∞,$\rho^{n+1}\frac{1-\rho^p}{1-\rho}$⟶ 0. Hence u_k(x) is uniformly convergent.

For the integral $\int_a^{\infty} f(x)dx$, if it is convergent, its value can be simply given by replacing the infinity with a natural number A that:

$$\int_a^{\infty} f(x)dx = \lim_{A\to\infty}\int_a^{A} f(x)dx. \tag{2}$$

Example 2.3: Calculate the integral: $\int_a^{\infty}\frac{dx}{x^2}$.

Choose a natural number b which is efficiently large to replace the infinity, then:

$$\int_a^{\infty}\frac{dx}{x^2} = \lim_{b\to\infty}\int_a^{b}\frac{dx}{x^2} = \lim_{b\to\infty}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{a} \tag{3}$$

Hence the value of the improper integral is known.

3. Method

Let's introduce some notations:

$$S_n(\mathrm{x}) = |\sum_{k=n}^{\infty} u_k(\mathrm{x})| \tag{4}$$

$$I_n(\mathrm{x}) = \int_n^{\infty} f(k)dk \tag{5}$$

Theorem 3.1 For all k∈N, let f(x) = u_k(x). The function f(x) is continuous and monotone between every interval [k, k+1], then the method below can be used to test the uniform convergence of series:

$$I_{n+1}(\mathrm{x}) \le S_{n+1}(\mathrm{x}) \le I_n(\mathrm{x}) \tag{6}$$

Proof:

According to the Cauchy's convergence test, to test whether S_{n+1}(x) is convergent or divergent is identical to test the uniform convergence of u_k(x).

Hence let's consider the interval [n, ∞] where the function f is defined on is being divided into unit subintervals [n, n+1], [n+1, n+2], …, [n+p-1, n+p], … for every p∈N.

Afterwards, the total sum of f(k), for every k ≥ n+1, actually is the S_{n+1}(x). Then:

$$S_{n+1}(\mathrm{x}) = |\sum_{k=n+1}^{\infty} f(k)| \tag{7}$$

Now using the improper integral can get the upper and lower bound of S_{n+1}(x).

$$\left|\sum_{k=n+1}^{\infty} f(k)\right| = S_{(n+1)(x)} \leq \int_{n}^{\infty} f(k)dk = I_n(\mathrm{x}) \tag{8}$$

On the other side:

$$I_{n+1}(\mathrm{x}) = \int_{n+1}^{\infty} f(k)dk \leq | \sum_{k=n+1}^{\infty} f(k) | = S_{n+1}(\mathrm{x}) \tag{9}$$

Combining (8) and (9) together, finally (6) is finished now.

When testing some function term series which is hard to be worked out through classical Weierstrass M-test, researchers can use this method and turn the series test into the improper integral to find out whether the improper integral is convergent or not.

4. Application

Next let's apply the method to a series that Weierstrass M-test can't solve it directly.

Example 4.1 [10]: When α > 0, please discuss the uniform convergence of $\sum_{n=1}^{\infty} x^{\alpha}\ e^{-nx}$ on [0, ∞].

Proof:

When 0 < α ≤ 1:

By the method, I_{n+1}(x) = $\int_{n+1}^{\infty} x^{\alpha}\ e^{-kx}dk = x^{\alpha-1}e^{-(n+1)x}$ ≤ S_{n+1}(x). Let x = $\frac{1}{n+1}$ and n → ∞. It is easy to conclude that I_{n+1}(x) is divergent. The series is divergent now.

When α > 1:

Using this method, I_n(x) = $\int_{n}^{\infty} x^{\alpha}\ e^{-kx}dk$ = $x^{\alpha-1}e^{-nx}$ ≥ S_{n+1}(x). Since α > 1, $x^{\alpha-1}e^{-nx}$ is convergent to 0 when choosing the n that is efficiently large.

Hence the series is uniformly convergent on [0, ∞].

Example 4.2: Showing that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+x}$ is uniformly convergent for x∈[0, ∞].

Proof:

It is easy to find that M-test doesn't apply on this integral. So, using the method above:

I_n(x) = $\int_{n}^{\infty} \frac{1}{k+x} dk = \lim_{b\to\infty} \ln\left(\frac{b+x}{n+x}\right)$ ≥ S_{n+1}(x). If choosing a natural number which is efficiently large, then S_{n+1}(x) ≤ I_n(x) < ε, for every ε > 0. Through this way the series is uniform convergence.

From the example it is obviously knowing that using improper integral to evaluate function series is helpful when Weierstrass M-test is not applicable.

5. Conclusion

The connection between improper integrals and infinite series has an inseparable relationship between their theory and application. When solving certain improper integrals, they can be transformed into infinite series summation. In this paper, a new method to bypass Weierstrass M-test and obtain uniform convergence is given and strictly proved. This method makes it possible to use improper integrals to determine the uniform convergence of function term series. In addition, by calculating the improper integral, the value of the function term series can be roughly estimated, which greatly facilitates approximate calculations in practical applications. But this method only applies when the function term series is positive. In the future, a method that can test all the function term series will be an expectation.

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