Comparative study of Riemann integral and Lebesgue integral in calculus

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Abstract. The aim of this research paper is to provide a comprehensive comparison between Riemann integral and Lebesgue integral. Integration is described as the inverse process of differentiation, which is used to determine the original function. Riemann integration is a specific type of definite integral applied to find the exact area under a function graph between two limits in a closed interval. Lebesgue integration, on the other hand, provides a more generalized framework for integration theory. Integrals are essential in mathematical modelling and analysis tools. Studying and comparing the similarities and differences between these two integration methods can help us better understand the essence and properties of integrals, so as to more accurately apply integration methods to solve practical problems. This paper provides a systematic analysis of the basics, definitions, concepts, and properties of Riemann integral and Lebesgue integral. Reasoning, proofs, and examples are consolidated to explain the properties and characteristics of these two integral methods. Finally, this paper explores the strengths and limitations of each integration methods, summarising their advantages and applicability to practical problems and providing insights into their respective computational methods and applicability in different contexts.

Keywords: Riemann integral, Lebesgue integral, comparison.

1. Introduction

In calculus, integration is the alternative process of differentiation meaning that it can be used to determine the original function f(x) by integrating its first derivative $f'(x)$ [1]. According to the first fundamental theorem of calculus, the differential of the area function under the curve in the close interval [a, b] is equal to the continuous function, showing the alternative nature in the relationship between differentiation and integration that could be expressed as [2]:

$$A'(x) = f(x), \qquad for\ all\ x \in [a, b]. \tag{1}$$

David stated that Leibniz refers integral calculus as "the calculus summitries, a name connected with the summa sign." And invented the integral sign $\int$ [3]. There are two types of integral--definite integral and indefinite integral. Indefinite integral is an integral that doesn't have upper and lower limit; in other words, it is a function that takes the antiderivative of another function which is presented as $\int f(x)\,dx$.

Definite integral is the integral with upper and lower limits, the symbol is $\int_b^a f'(x)\,dx$, when x is restricted to lie on the real line, the definite integral is known as a Riemann integral [4].

The Riemann integral is applied for determining the exact area under the function graph between two limits [5], a, and b, which they form a closed interval where $a < b$, besides, taking any points in this close interval which could be represented as [6]:

$$a = x_0 < x_1 < \cdots < x_n = b \tag{2}$$

These points split $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, where $i = 1, 2, \ldots, n$. Then, straight lines $x = x_i$, where $i = 1, 2, \ldots, n\text{-}1$ divide the area under a function into n trapezoid with curved edge. After that, take a point ξ_i on each subinterval and form small rectangle with the height $f(\xi_i)$ and width $x_i - x_{i-1}$. As $n \to \infty$, the sum of those small rectangles, also known as the Riemann sum, will approach to the exact area S under the function, meaning that the Riemann sum will be approximately equal to S [7]:

$$S \approx \sum_{i=1}^{n} f(\xi_i)\Delta x_i, \qquad (\Delta x_i = x_i - x_{i-1}) \tag{3}$$

Riemann integration and Lebesgue integration are the two most prevalent approaches for defining integrals. While Riemann integration has been extensively studied and widely taught in calculus courses, Lebesgue integration offers a more generalized framework that extends the scope of integration theory.

Riemann integration has several limitations [8], it cannot be applied to integrating unbounded or highly discontinuous functions. While the Lebasque integral can be used for a wider range of functions and has better mathematical properties than the Riemann integral [9]. Compared with Riemann integral, Lebesgue integration divides the value range of a function with different partitions and measures which means use Lebesgue measurable sets to replace all the intervals of Riemann integral. According to the concept of measure, Lebesgue integration is the generalization of the concept of measure on the plane, and it is established based on the measure theory. It defines the function on a more general set of points, not limited to [a, b], and this theory can deal with the situation of Bounded function and unbounded function [10].

Comparing the Riemann integral and the Lebesgue integral is significant and relevant for a deeper understanding of the nature and properties of integrals. Both methods of integration play an essential role in mathematics and applications, so comparing them helps provide us with a deeper exploration of the concept of integration.

Furthermore, by comparing the definitions, properties, and computational methods of these two integral methods, it is possible to identify the differences and commonalities between them, leading to a better understanding of integrals. Besides, as these two methods have different applicability when dealing with different types of functions and solving practical problems, comparing them could also enable us to explore the applicability of integral methods.

2. Methodology

2.1. Theoretical analysis
Thorough study and comparison of the definitions of the Riemann and Lebesgue integrals will be carried out. Their basics, the way they are defined, and related concepts and properties will be carefully explored.

The differences and commonalities in the mathematical theory of these two integration methods will be systematically analyzed, in particular, their applicability and consistency of results under different classes of functions. Then, clarify and explain the properties and characteristics of these two integral methods by means of logical reasoning, proofs, and examples.

2.2. Numerical calculations
Compare and evaluate numerical computation methods for the Riemann and Lebesgue integrals. The study includes, but is not limited to, aspects of partitioning methods, numerical approximation methods, and computational efficiency.

Design and implement numerical computation experiments to compare the precision and computational efficiency of these two integration methods in different situations by using simple but effective numerical approximation techniques such as the rectangular method or the trapezoidal method.

Execute numerical computation experiments and analyze the results using appropriate calculation tools and software to validate theoretical analyses and draw reliable conclusions.

2.3. Application comparison
In this study, there will be exploration of the differences in the application of Riemannian and Lebesgue integrals to real-world problems and a comparison of their strengths and applicability in dealing with specific problems.

2.3.1. Selection of application cases. Select some representative application cases from the fields of computational mathematics, physics, and engineering. These cases should be able to cover different types of functions and problems so that the differences in the applications of the Riemann and Lebesgue integrals in different situations can be thoroughly compared.

2.3.2. Case analysis. Each application case will be analysed in detail, including the general context of the problem, the specific problem to be solved, and the mathematical models and functions involved.

For each case, the Riemann Integral and Lebesgue Integral will be applied separately and their effectiveness in solving the problem and the ease of application will be compared.

In the analysis process, attention will be paid to observing the ability of the Riemann and Lebesgue integrals to deal with different types of functions, discontinuous functions, and with oscillations of functions.

2.3.3. Evaluation of comparative results. Compare the results obtained by the Riemann integral and Lebesgue integral in each application case. The accuracy, feasibility, and reliability of the results will be assessed.

Analysing the strengths and limitations of each integration method and how well it can be applied to specific problems, we will consider the geometric intuition and computational feasibility of the Riemann integral, as well as the advantages of the Lebesgue integral in dealing with functions of a specific nature.

2.3.4. Results discussion. Comparatively, analyse the differences in applying the Riemann and Lebesgue integrals in each application case, and summarise their advantages and applicability to practical problems.

Discuss the applicability of each integration method in dealing with different types of problems and provide advice on choosing the appropriate integration method for a particular problem. Highlight the areas of application and limitations of Riemannian and Lebesgue integrals in practical problems and provide potential directions for further research.

3. Main body

3.1. Theoretical analysis
Riemann integral, also known as the definite integral. Generally speaking, Riemann integral divides the area into many trapezoids of different sizes. By adding all these areas of the trapezoids, an approximation can be got and that is the integral. Originally, Riemann integral can be considered as the development based on the Newton's integral and Cauchy's integral, and that is the basis of integral calculus.

Strictly defined, the interval [a, b] in the Riemann integral is arbitrarily inserted n-1 points, and [a, b] is divided into n cells, and this [a, b] divides T. So the modulus of partition T shows the fineness of partition T. This show that Riemann integral is defined by dividing intervals and summing the function values on each subinterval.

Unlike the way Riemann integral is defined, Lebesgue integral is defined by measuring the relationship between a function and a specific measure. In Lebesgue a function maps X to R is called countable simple if it has only a countable number of values. Given a measure space (X, M) the integral of a nonnegative measurable countable simple function f is defined by $\int f dx = \sum_{i=1}^{\infty} a_i \mu(f^{-1}\{a_i\})$.

Going further, assume that f is a measurable function, define $f^+ := max(f, 0) \geq 0$ and $f^- := -min(f, 0) \geq 0$. Thus, let $f = f^+ - f^-$, define $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$. Here at least one term is finite so that the integral exits. And could be the definition for all arbitrary.

3.2. Numerical calculations

The Riemann integral is defined as the limit of Riemann sums of a function as the partitions of the interval go to zero. Essentially, it approaches the area under the curve by dividing it into a series of rectangles and then summing their areas.

The Lebesgue integral approaches the problem differently. Instead of dividing the interval on the x-axis, it divides the range of the function on the y-axis. It then sums up the areas of these 'horizontal' strips. This approach handles a broader class of functions and is better suited to handling limits and infinite series.

The Riemann integral is the standard used in numerical methods, such as the Trapezoidal Rule or Simpson's Rule (easy to find a specific numerical example/implemented code). Lebesgue integration, on the other hand, is more abstract and is not typically used in numerical computation. The Lebesgue integral is more about theoretical underpinnings than practical computation.

An example of a function that is Lebesgue integrable but not Riemann integrable is the characteristic function (also known as the indicator function) of the rational numbers in [0, 1]. If x is rational, f(x)=1. If x is irrational, f(x)=0.

This function is not Riemann integrable on [0, 1] because, no matter how you choose your partition, every subinterval will contain both rational and irrational numbers, so you will not be able to create a well-defined Riemann sum (the infimum and supremum of each subinterval will be 0 and 1, respectively).

However, the function is Lebesgue integrable. The Lebesgue integral of over [0, 1] is 0, because the measure of the rational in [0, 1] is 0 (the set of rational numbers in [0, 1] is countable, and in the Lebesgue measure, any countable set has measure 0). The function f is equal to 1 on a set of measure 0 and 0 almost everywhere else, so its Lebesgue integral is 0.

This example illustrates how the Lebesgue integral is capable of handling more "problematic" functions than the Riemann integral, making it a powerful tool in analysis and probability theory.

3.3. Application comparison

3.3.1. Calculating the area of a region for function $f(x) = x^2$ on the closed interval [-2,2]. The function type used for comparison is a continuous function of a closed interval [a, b]. The accuracy of calculating the exact area will be compared, at the same time the simplicity and ease of calculation will also be considered.

The graph of function $f(x) = x^2$ is presented below (figure 1); since it is continuous on [-2,2], both Riemann and Lebesgue integrals are applicable.
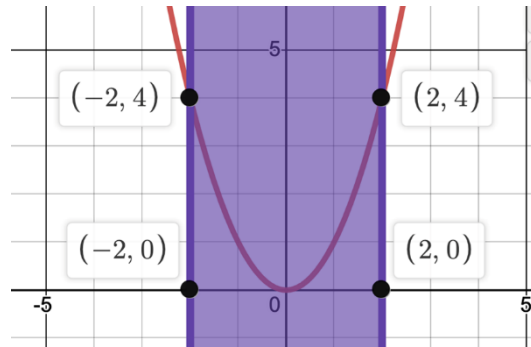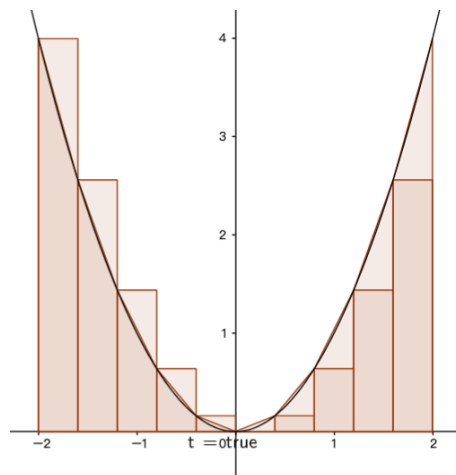
Figure 1. Graph of function $f(x) = x^2$ in the interval of [-2, 2]

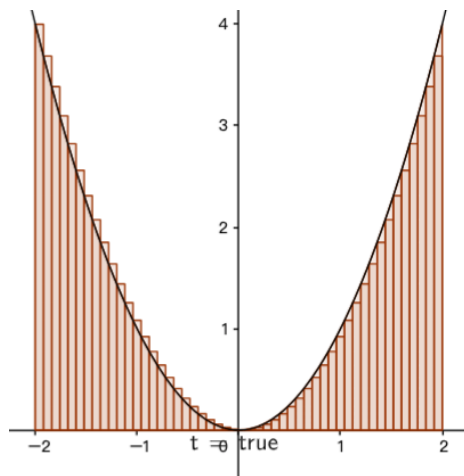Sample of calculation for n=10 (figure 2) using Riemann integral method:

$$\int_{-2}^{2} x^2 dx \approx \frac{2-(-2)}{10}\left(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_8) + f(x_9)\right) = 5.44 \qquad (4)$$

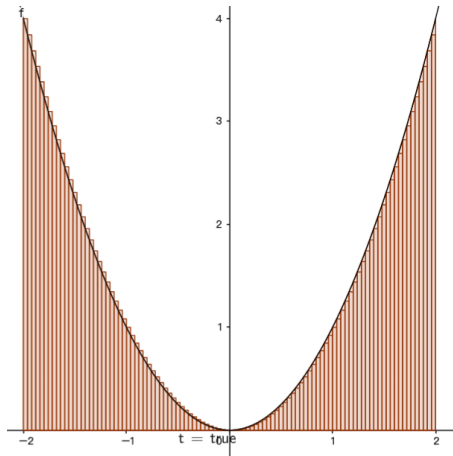Presented value of trapezoid sum with different number of subintervals:

Figure 2. Graph of subintervals=10, Value of trapezoid sum=5.440.

The area under the in the interval of [-2,2] is covered by 10 rectangles.
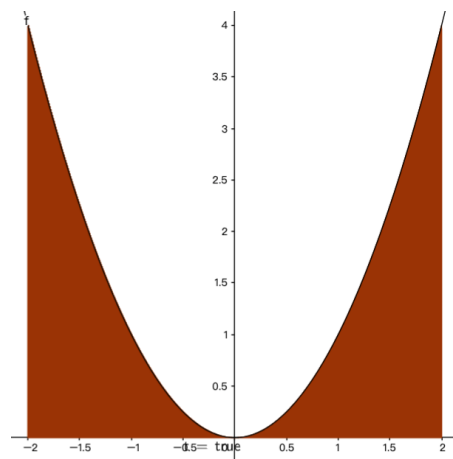

Figure 3. Graph of subintervals=50, Value of trapezoid sum=5.338.

The area under the in the interval of [-2,2] is covered by 50 rectangles.

Figure 4. Graph of subintervals=100, Value of trapezoid sum=5.334.

The area under the in the interval of [-2,2] is covered by 100 rectangles.

Figure 5. Graph of subintervals=500, Value of trapezoid sum=5.333.

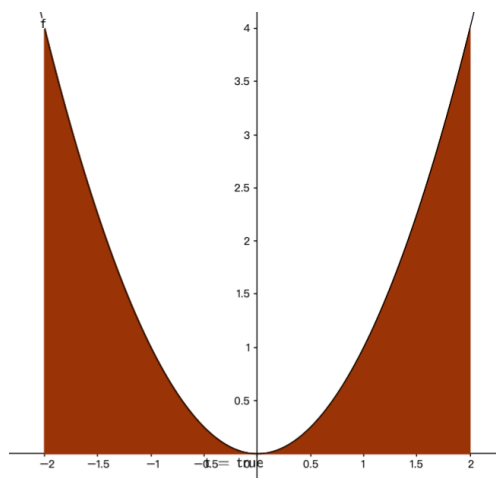The area under the in the interval of [-2,2] is covered by 500 rectangles.

Figure 6. Graph of subintervals=1000, Value of trapezoid sum=5.333.

The area under the in the interval of [-2,2] is covered by 1000 rectangles.

In conclusion, as n tends to infinity, the calculated area under the function using Riemann integral tends to 5.33(figure 6).

However, when Lebesgue integral is applied to solve the exact area, the steps will be: Construct a sequence $f_n := \sum_{k=1}^{2^n}(\frac{k-1}{2^n})^2 \times XA_{n,k}$, with $A_{n,k} := \{\frac{k-1}{2^n} \leq x < \frac{k}{2^n}\}$. Since $f_n \leq f_{n+1}$ and $f_n(x) \to f(x) = x^2$ pointwise for $x \in [-2,2]$, by the monotone convergence theorem,

$$\int_0^2 x^2\, dx = \int_0^1 f = \lim_{n\to\infty}\int_0^1 f_n = \lim_{n\to\infty}\sum_{k=1}^{2^n}\frac{(k-1)^2}{2^{2n}}\left(\frac{k}{2^n} - \frac{k-1}{2^n}\right) = \frac{\lim_{n\to\infty}1}{2^{3n}}\sum_{k=0}^{2^n-1}k^2 = \frac{16}{3} \qquad (5)$$

Apply antiderivative to the function $f(x) = x^2$ in the interval of [-2,2]):

$$\int_{-2}^2 x^2\, dx = \left[\frac{1}{3}x^3\right]_{-2}^2 = \frac{1}{3}\times 8 - \left(\frac{1}{3}\times -8\right) = 5.3333\ldots \approx 5.333(4sf) \qquad (6)$$

In evaluating the Lebesgue integral and the Riemann sum for the function $f(x) = x^2$ on the interval [-2, 2], it is observed that both methods provide approximations of the area that are very much like the actual area under the curve. At the same time, it is notable that Lebesgue integrals are usually used for more complex functions, while Riemann integrals rely on splitting the area into subintervals.

For Riemann integration, the accuracy of the Riemann sum is achieved by choosing a sufficiently granular partition that allows the behavior of the quadratic function to be captured within each subinterval. This choice allows the Riemann sum to converge to the exact area value under the curve. Well-chosen splits and consideration of smaller subintervals help to allow the Riemann sum to converge to the exact area value.

On the other hand, Lebesgue integral's strength lies in the ability of the Lebesgue measure to capture the complex structure of a function. By defining the integral as the limit of a sequence of simpler functions, the Lebesgue integral effectively applies to functions with a variety of complexity levels.

The dual accuracy of Lebesgue and Riemann integration methods in calculating the area under the curve of the function $f(x) = x^2$ highlights the soundness and suitability of these integration techniques when being applied to continuous functions such as $f(x) = 5x^3 + 2x^2 + 4x + 7$, and so on. This finding is crucial for precise integration techniques for problem solving in fields such as physics, engineering, and computer science but limited to continuous function only.

Although both integrations yield the same results for this particular continuous function, their computational complexity is quite different, with the Riemann integral being considerably more straightforward to compute than the Lebesgue. And, for more complex functions or different intervals, the performance of the two methods may differ; thus, the computational efficiency of the two methods remains an important practical consideration when dealing with larger and more complex functions.

4. Conclusion

This research paper set out to provide a comparative analysis of Riemann integral and Lebesgue integral from a mathematical perspective. It explores their definitions, mathematical properties, applications, and computational methods. In the main body, the differences and similarities between Lebesgue and Riemann integration methods for evaluating the area under the curve of a function are discussed. Both methods provide approximations of the area that closely resemble the actual area. Riemann integration involves splitting the interval into subintervals and capturing the behavior of the function within each subinterval to achieve accuracy. On the other hand, Lebesgue integration's strength lies in its ability to handle complex functions by defining the integral as the limit of a sequence of simpler functions. The results of this study indicate that the performance of the two methods may vary for complex functions or different intervals, therefore considering computational efficiency becomes important when dealing with larger and more complex functions. Overall, this study strengthens the idea that integrals play a fundamental role in mathematical analysis, calculus, and various areas of science and engineering,

highlighting the soundness and suitability of Lebesgue and Riemann integration methods for precise integration techniques in mathematics, physics, engineering, and computer science.

Authors Contribution
All the authors contributed equally, and their names were listed in alphabetical order.

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