Relativistic path integrals

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Abstract. Classical and Quantum mechanics are the two milestones of physics and mathematics. The path integral describes the generalised form of action from classical to quantum mechanics. This paper has reviewed some fundamental concepts and results in classical dynamics and quantum mechanics. The research method of the whole project is mainly theoretical derivations of applied mathematics and mathematical physics. This paper provides different perspectives to investigate the applications of path integrals. This paper also builds a connection between path integrals and the Unruh temperature.

Keywords: path integrals, simple harmonic, oscillator, relativistic quantum, mechanics, unruh effect, detector models.

1. Introduction

Quantum mechanics is normally introduced and formulated by the Schrödinger equation involving Hamiltonian operators. The quantum counterpart of the classical Hamiltonian is the operator. Feynman developed another formulation of quantum mechanics based on the Lagrangian based on Dirac's work [1]. Here one considers the phase with

$$\exp\left(\frac{i}{\hbar}S\right)=\exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2}Ldt\right) \tag{1}$$

where t_1 and t_2 are initial and final time. And all possible paths are summed over with fixed initial and final boundary conditions. This path (or 'sum') can be thought as a 'kernel' which is also an answer to the Schrödinger's equation.

Usually, the path integral approach is not as effective as applying the Schrödinger equation directly. However, it is particularly suitable for relativistic problems to use path integrals. This is because a Hamiltonian treatment singles out time whereas space and time can be treated on an equal footing through a Lagrangian. Relativistic quantum field theories have also proved that path integrals is useful in that study. A fascinating route is given in the Bailin and Love's book [2], which begins with the Gaussian integral

$$\int_{-\infty}^{\infty}\mathrm{dy}\exp\left(-\frac{1}{2}\mathrm{ay}^2\right)=(2\pi)^{1/2}\mathrm{a}^{-1/2}, \tag{2}$$

and develops in order the theory of quantum field theory and path integrals.

This paper has studied the path integrals from the derivations towards the applications [1, 3]. First, the path integral formulation will be introduced in non-relativistic quantum mechanics. Readers will try to study it in a basic application, which is simple harmonic oscillator. Readers will also understand in detail some other applications of the path integrals with relativistic problems, such as the accelerating detector and the detector in heat bath. Furthermore, readers will study the connection with a few interesting predictions, for example, the Unruh temperature.

2. Path integrals in quantum mechanics

2.1. Hamiltonian and lagrangian formulation

Schrödinger developed a formulation in terms of the wave functions paralleled to Heisenberg's formulation of quantum mechanics[4]. He was also inspired by de Broglie's and Einstein's ideas of matter waves and wave-particle duality. In mathematical form, the Schrödinger equation is formulated as [3]

$$\mathrm{H}\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial \mathrm{t}}, \tag{3}$$

where H is the Hamiltonian, $\psi$is the wave function, the reduced Planck constant is $\hbar = h/(2\pi)$ and $h \approx 6.626 \times 10^{-34}Js$ is the Planck constant. The Hamiltonian is defined as the total mechanical energy of the system and is always in the form of

$$\mathrm{H}(\mathrm{x},\mathrm{p},\mathrm{t}) = \mathrm{T} + \mathrm{V} = \frac{\mathrm{p}^2}{2\mathrm{m}} + \mathrm{V}(\mathrm{x},\mathrm{t}), \tag{4}$$

where L is the Lagrangian, T and V are the kinetic and the potential energy.

The Lagrangian of a dynamical system is a function which describes the state of the whole dynamical system. It is given in equations as

$$\mathrm{L}(\mathrm{x},\dot{\mathrm{x}},\mathrm{t}) = \mathrm{T} - \mathrm{V} = \frac{1}{2}\mathrm{m}\dot{\mathrm{x}}^2 - \mathrm{V}(\mathrm{x},\mathrm{t}). \tag{5}$$

In classical mechanics[5], when considering a particle (of mass m) moving with respect to a conservation force, the force gives

$$\mathrm{F}(\mathrm{x_i},\mathrm{t}) = -\nabla\mathrm{V}(\mathrm{x_i},\mathrm{t}) \tag{6}$$

we have the equation in one dimension

$$\mathrm{F}(\mathrm{x},\mathrm{t}) = -\frac{\partial\mathrm{V}(\mathrm{x},\mathrm{t})}{\partial\mathrm{x}}. \tag{7}$$

From Newton's 2nd law

$$\mathrm{F} = \mathrm{m}\ddot{\mathrm{x}}, \tag{8}$$

the relation is rewritten as

$$-\frac{\partial\mathrm{V}(\mathrm{x},\mathrm{t})}{\partial\mathrm{x}} = \mathrm{m}\ddot{\mathrm{x}} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial\mathrm{T}}{\partial\dot{\mathrm{x}}}\right). \tag{9}$$

Because Tand V are independent of x and $\dot{\mathrm{x}}$ respectively in the Lagrangian, the equation can be written as

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial\mathrm{L}}{\partial\dot{\mathrm{x}}}\right) = \frac{\partial\mathrm{L}}{\partial\mathrm{x}}, \tag{10}$$

this is also called an Euler-Lagrangian equation.

Moreover, the action S is always defined as an integral between two particularly fixed events $\mathrm{x_1}(\mathrm{t})$ and $\mathrm{x_2}(\mathrm{t})$ linkd by the path $\mathrm{x}(\mathrm{t})$, which can be written as

$$S = \int_{t_1}^{t_2} L(x,\dot{x},t)dt. \tag{11}$$

2.2. *The path integral*

The Hamilton's principle describes that the line integral (or called the time-evolution) of a system from t_1 to t_2 is the action integral

$$S = \int_{t_1}^{t_2} L dt \tag{12}$$

which is stationary regardless of different coordinates.

When studying a moving particle from position a to b, there are many possible paths. Each path has action $S[x(t)]$. The square brackets also mean that S is a functional of x. The phase is considered in Feynman and Hibbs's book [1]

$$\exp\left(\frac{i}{\hbar} S\right) = \exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2} L dt\right) \tag{13}$$

which is also considered as a contribution from a particular path to the propagator. All possible trajectories can be then summed over with fixed boundary values and conditions [6]. The sum can also be formally written as

$$K = \int_a^b \exp\left(\frac{i}{\hbar} S\right) Dx(t) \tag{14}$$

which is called a path integral. In the path integral, there can be some smooth functions [7]. In the next part, an example of the path integrals is given to study the application of how it can be used.

2.3. *Application: the simple harmonic oscillator*

In this model, some conditions should be defined at the start. The initial and final time can be defined respectively. And it is easier to set $t_1 = 0$ and $t_2 = T$, where T is also the time interval between the two events. The position function of the particle is

$$x = x_i + \frac{(x_f - x_i)}{T} t + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n t}{T}\right) \tag{15}$$

where x_i and x_f is defined to represent the initial and the final positions, and b_n is fixed to determine the trajectories (paths). The action of the particle is now

$$S = \int_0^T \left[\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2\right] dt, \tag{16}$$

and in the square bracket it is the Lagrangian. In the process of evaluating the exponential term for this path integral, the difficult step is to calculate the integral

$$\int_0^T x^2 dt \tag{17}$$

where

$$\begin{aligned} x^2 = \quad & x_i^2 + \frac{\left(x_f - x_i\right)^2}{T^2} t^2 + \sum_{n=1}^{\infty} b_n^2 \sin^2\left(\frac{\pi n t}{T}\right) + \\ & 2\left[x_i + \frac{x_f - x_i}{T} t\right] \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n t}{T}\right) + 2 x_i \frac{(x_f - x_i)}{T} t + o, \end{aligned} \tag{18}$$

And the action after simplification is

$$\begin{aligned} S = \quad & \frac{1}{2} m \frac{\left(x_f - x_i\right)^2}{T} + m \sum_{n=1}^{\infty} \frac{b_n^2 \pi^2 n^2}{4T} \\ & -\frac{1}{2} m \omega^2 \left[T \frac{\left(x_f^2 + x_i^2 + x_i x_f\right)}{3} + \sum_{n=1}^{\infty} \left[\frac{T}{2} b_n^2 + \frac{T}{n\pi}\left(x_i - x_f(-1)^n\right) b_n\right]\right] \end{aligned} \tag{19}$$

In this case, the integral can be rewritten as

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$$\int_{-\infty}^{\infty}\prod_{i=1}^{\infty} db_n e^{\frac{i}{\hbar}S}, \tag{20}$$

and the exponential term is given by

$$\exp\left(\frac{i}{\hbar}S\right) = \exp\left(\frac{i}{\hbar}\left[\frac{m(x_f - x_i)^2}{2T} + \frac{1}{6}m\omega^2 T\left(x_f^2 + x_i^2 + x_i x_f\right)\right]\right)$$
$$\times \prod_{n=1}^{\infty} \exp\left(\frac{i}{\hbar}\left[b_n^2\left(\frac{m\pi^2 n^2}{4T} - \frac{Tm\omega^2}{4}\right) - \frac{1}{2}b_n m\omega^2\frac{T}{n\pi}\left(x_i - x_f(-1)^n\right)\right]\right). \tag{21}$$

Readers should noticed that this is the Gaussian integral formula with imaginary instead of real coefficients. Hence, the formula for the integral is rewritten as

$$\int_{-\infty}^{\infty} e^{iax^2+ibx}dx = \sqrt{\frac{\pi i}{a}}e^{-\frac{ib^2}{4a}}, \tag{22}$$

where $a, b \in C$. And the coefficients are

$$a = \frac{1}{\hbar}\left(\frac{m\pi^2 n^2}{4T} - \frac{Tm\omega^2}{4}\right) \tag{23}$$

and

$$b = -\frac{1}{\hbar}\frac{1}{2}m\omega^2\frac{T}{n\pi}\left(x_i - x_f(-1)^n\right). \tag{24}$$

Hence, the path integral is solved for the simple harmonic oscillator.

3. Relativistic quantum mechanics

3.1. Klein-gordon equation

The Lagrangian of the form $L = T - V$ can describe conservative forces. For a free particle in the relativistic form, the motion of moving particle in one dimension form is formulated by the Lagrangian [8]

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}. \tag{25}$$

In mathematics, Lagrangian under the Legendre transform can generate Hamiltonian. Hence, the relativistic Hamiltonian can be obtained

$$H = \sqrt{m^2c^4 + p^2c^2}. \tag{26}$$

The relativistic wave equation can be derived by the method for the non-relativistic free particle wave equation

$$\sqrt{m^2c^4 - c^2\hbar^2\nabla^2}\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{27}$$

where the RHS can be also represented by

$$E \longrightarrow +i\hbar\frac{\partial}{\partial t}. \tag{28}$$

Now, introduce the four-momentum,

$$p = \left(\frac{E}{c}, \mathbf{p}\right), \tag{29}$$

Where p is the linear momenta of the particle. The important step is to square the energy term in (28), which gives

$$E^2 \longrightarrow -\hbar^2\frac{\partial^2}{\partial t^2} \tag{30}$$

and it is also true for

$$E^2 = m^2c^4 - \hbar^2c^2\nabla^2 \tag{31}$$

Then the differential equation is linearised. Hence, the relativistic wave equation can now be written as

$$(m^2c^2 - \hbar^2\nabla^2)\psi = -\frac{\hbar^2}{c^2}\frac{\partial^2\psi}{\partial t^2}, \tag{32}$$

or without the bracket

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0 \tag{33}$$

Moreover, a new four-dimensional Laplacian can be introduced, which is

$$\Box = \partial^\mu\partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{34}$$

Hence, the equation can be simplified as

$$\psi + \frac{m^2c^2}{\hbar^2}\psi = 0, \tag{35}$$

which is a form of the Klein-Gordon Equation. Also, there is another derivation in geophysical fluid mechanics towards the Klein-Gordon Equation [9]. There are some further applications and discussions of the Klein-Gordon Equation. The superseded version of the Klein-Gordon equation can be derived. Paul Dirac discovered the Dirac equation in 1928 and expanded the solutions to the higher dimensions [10].

3.2. *Euler-lagrangian field equation*

This subsection is aimed to give a taster in field theory and link it with path integrals. Now considering the Klein-Gordon equations, with Planck units

$$\psi + m^2\psi = 0. \tag{36}$$

Recall that the formula of action is

$$S = \int Ldt \tag{37}$$

where

$$L = L(q, \dot{q}, t) \tag{38}$$

Also, another formula given in Landau and Lifshitz's book[8] is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi \tag{39}$$

where $\mathcal{L}$ is the Lagrangian density and ϕ is the classical scalar field. Hence in the field theory version, the action becomes

$$S = \int \mathcal{L}dtdxdydz \tag{40}$$

where

$$\mathcal{L} = \mathcal{L}\left(\phi, \partial_\mu\phi, x^\mu\right) \tag{41}$$

Applying the Euler-Lagrangian equation, the Euler-Lagrangian field equation is

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{42}$$

where

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$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi, \tag{43}$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi. \tag{44}$$

Finally, the integral is rewritten as

$$\int D\phi e^{\frac{i}{\hbar}S}. \tag{45}$$

Furthermore, the example can be expanded to 1+1 dimension, the Lagrangian density can be rewritten as

$$\mathcal{L} = \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\partial_1\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{46}$$

4. Application: unruh effect and the detector models

4.1. Unruh temperature

The Unruh effect is that the blackbody radiation can be observed by an accelerating observer, but cannot be observed by an inertial observer[11]. In other words, a non-zero temperature without other temperature sources can still be measured by a thermometer on an accelerating observer.

William Unruh derived the Unruh temperature in 1976 [12], and it is the temperature experienced by an accelerating observer in the vacuum field. The Unruh temperature is defined as

$$T = \frac{\hbar a}{2\pi c k_B}, \tag{47}$$

where a is acceleration, c is the speed of light in the vacuum field, and $k_B \approx 1.38 \times 10^{-23}\ m^2 kgs^{-2}\ K^{-1}$ is called the Boltzmann constant. This is also worthy to mention that Stephen Hawking derived the Hawking temperature in 1974[13] has the same value and representation as the Unruh temperature.

4.2. Detector models

S.W.Hawking and Werner Israel's book [14]gave me motivation for the following part of the project. The key assumption is that'Can temperature be generated the acceleration?' In this work, the exact propagator were avoided from being computed to ease the calculation. It will be also easier to use Planck units, consider the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x) + xj(t), \tag{48}$$

Where $j(t)$ is the source term. The source term $j(t)$ can also be understood as the driving force and it is a function of time t. Mathematicians and physicists always understand the detector model using the source term in the path integrals. Hence, the Euler-Lagrangian equation becomes

$$\frac{d}{dt}(m\dot{x}) = V'(x) - j(t). \tag{49}$$

In section 3, the Lagrangian density was introduced. Then, using the same method in D.Bailin and A.Love's book [2], the path integral to the model can be rewritten as

$$\int D\phi e^{iS(\phi)+i\int j(x)\phi(x)}, \tag{50}$$

where the source term j(x) is

$$j(x) = \int d\tau Q(\tau)\delta^4(x - x(\tau)) \tag{51}$$

In the following parts, different performances of the models will be introduced in order to study the link between the path integrals and the Unruh effect.

4.2.1. Accelerating detector. Firstly, the trivial model is thhe stationary detector under zero temperature. And the coordinates in the four-vector are respectively

$$x^0(\tau) = \tau \tag{52}$$

$$x^1 = x^2 = x^3 = 0. \tag{53}$$

For the rest of this subsection, the accelerating detector is taken into account at zero temperature. Introduced in D.Bailin and A.Love's book [2], the functional W is

$$W = \int_{-\infty}^{\infty} d\tau \int_{\infty}^{\infty} d\tau' Q(\tau)Q(\tau')f(\tau-\tau') \tag{54}$$

where the path integral f is

$$f(\tau) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik^0\tau}}{(k^0)^2-\mathbf{k}^2+i\epsilon-m^2} \tag{55}$$

Here the error is fixed by ϵ term and the integral is now well defined.

In J.R. Anglin's article [15], the trajectory of the detector is considered by

$$\begin{aligned} x^0 &= \frac{1}{a}\sinh(a\tau) \\ x^1 &= \frac{1}{a}\cosh(a\tau) \\ x^2 &= 0 \\ x^3 &= 0 \end{aligned} \tag{56}$$

In the four-vector form, and the scalar product is that

$$\mathrm{p}^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2, \tag{57}$$

and

$$\mathrm{k}\cdot\mathrm{x} = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3. \tag{58}$$

So in the path integral, the exponential term can be written as

$$e^{-ik^0(x^0-x^{0\prime})+ik^1(x^1-x^{1\prime})}. \tag{59}$$

From the result in Anglin's paper[15], the detector's trajectory can be written as

$$\begin{aligned} x^0 &= \frac{1}{a}\left(\sinh(a\tau) - \sinh(a\tau')\right) \\ x^1 &= \frac{1}{a}\left(\cosh(a\tau) - \cosh(a\tau')\right) \end{aligned} \tag{60}$$

Now operate the four-vector formula to calculate x, which is

$$(x^0)^2 - (x^1)^2 = -\frac{2}{a^2} + \frac{2}{a^2}\cosh\left(a(\tau-\tau')\right) \tag{61}$$

Then we have

$$(x^0)^2 - (x^1)^2 = \frac{4}{a^2}\sinh^2\left(\frac{1}{2}a(\tau-\tau')\right). \tag{62}$$

Then the Lorentz invariance becomes

$$\tau-\tau' \longrightarrow \frac{2}{a}\sinh\left(\frac{1}{2}a(\tau-\tau')\right). \tag{63}$$

$\tau-\tau' = 0$ is for the stationary model and for the accelerating model $\tau-\tau'$ becomes

$$(\tau-\tau')^2 \longrightarrow \frac{4}{a^2}\sinh^2\left(\frac{1}{2}a(\tau-\tau')\right). \tag{64}$$

The hyperbolic term has the property

$$\tau \longrightarrow \tau+\frac{4\pi}{a}i, \tag{65}$$

which means that the period for the hyperbolic term is $(4\pi)/a$, and remember that when this term is squared the negative sign is taken away. Thus, the period for the whole trajectory is

$$\tau \longrightarrow \tau+\frac{2\pi}{a}i. \tag{66}$$

And the value of the period is exactly the same as inverse value of Unruh temperature $a/(2\pi)$ with Planck units.

4.2.2. Detector in heat bath. When studying the model under zero temperature, the fraction in (55) is

$$\frac{i}{(k^0)^2-\mu^2+i\epsilon} \tag{67}$$

where $\mu^2 = k^2 + m^2$. This representation can also be expanded in a finite temperature in a heat bath. With this idea in Ashok Das's paper [16], the fraction term is rewritten as

$$\frac{i}{(k^0)^2-\mu^2+i\epsilon}\frac{e^{\beta|k^0|}}{e^{\beta|k^0|}+1}+\frac{i}{(k^0)^2-\mu^2-i\epsilon}\frac{1}{e^{\beta|k^0|}+1}, \tag{68}$$

where $\beta = 1/(k_B T)$ is inversely proportional to the temperature (consider the conservation of dimensions and note that $k_B T$ is in the unit of energy). When β tends to infinity the temperature tends to 0, which proves that the model can also become the same form as it in the zero temperature conditions. Alternative approach can be using power series.

Anglin claims in his article[15] that the effect and contribution of the heat bath under the temperature $kT = \frac{\hbar a}{2\pi c}$ is the same as the effect of an accelerating detector in the scalar vacuum field.

5. Conclusion

This paper has seen an overview of the path integral. From this paper, we can make a useful connection between temperature and acceleration using path integrals, especially in the Unruh effect.

At the end of section 2, the solution and the coefficients can be simplified using some other methods. The author tried to reach the same result as Anglin's for the path integrals under the heat bath [15], but the calculation is interrupted when I meet the quarter circle in the contour integral for the k^0 term. Other creative methods could be to use the Bessel function as Anglin.

There are some possible applications that might be related to this paper. For example, the Buchdahl limit is a nice discussion in the area between path integrals and thermodynamics [17]. I will be more interested in studying in this branch in the future.

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