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# European option pricing based on FSDE driven by fractional Brownian motion

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**Abstract.** In the actual financial market, the classical Black-Scholes (B-S) model can't perfectly describe the process of stock price. Besides, memory effect is an important phenomenon in financial systems. Thus, in this paper, we establish a fractional order stochastic differential equations (FSDE) which is driven by fractional Brownian motion (fBm) to describe the effect of noise memory and trend memory in financial pricing. Finally, we derive a European option pricing formula based on the established model. After conducting an empirical analysis based on the SSE 50ETF, we find that the established model performs better than the traditional one.

**Keywords:** European option pricing, Fractional stochastic differential equations, Fractional Brownian motion, Hurst index, Empirical research

## 1. Introduction

Hurst, a statistician, first proposed that time series have long memory, and proposed to use $R/S$ analysis method to test the long memory characteristics of event series [1]. Then there are many scholars pointed out that asset price fluctuations exhibit long memory [2-6]. Hurst index can be used to measure whether time series have long memory. When $H = 0.5$, time series can be described by random walks, and time series is the <ion. However, when $0.5 < H < 1$, time series has positive correlation and persistent behavior, which is long-dependence memory. In the case of $0 < H < 0.5$, time series has negative correlation and anti-persistent behavior, which is called short-dependence memory. In 1968, Mandelbrot and Van Ness put forward fractional Brownian motion [7]. For time series with long memory, the mathematical model combined with Hurst index forms a complete and self-consistent research system, which enables people to study how long memory affects the change of time series. In 1994, Peters applied Hurst index and fractional Brownian motion to capital market, pointed out that stock price series obey fractional Brownian motion, and proposed the famous fractal market hypothesis [8].

A large number of scholars have studied option pricing based on fractional Brownian motion. Based on fractional Itô integration, Necula (2002) used the risk-neutral valuation theorem to obtain the Black-Scholes pricing formula and analytical solution [9]. The Monte Carlo simulation method was used by Wang,J. et al.(2021) to price the Equity-Linked Securities option by fractional Brownian motion, and the new model produces a great deviation[10]. Liu,Z.B. and Huang,S. (2021) established a model based on GARCH and fBm so as to provide reference for China's upcoming carbon option trading through carbon option price forecasting research[11]. Zhao,P. et al.(2022) presented a new pricing method on N-fold compound option by adopting the theory of fuzzy sets into a fractional stochastic financial model [12]. Dufera,T.T. (2024) examined the impact of fBm on option pricing and dynamic delta hedging and ultimately the dynamic interaction between the Hurst index and option pricing is found, which provides valuable implications for effective risk management strategies [13].

However, the memory effect includes noise memory effect and trend memory effect. Stochastic differential equations driven by fractional Brownian motion can only describes the noise memory but cannot be used to study the trend memory effect of stock price. So Li (2014) proposed that a fractional order stochastic differential equation driven by Brownian motion could be used to solve this problem [14].

This is due to the fact that fractional integral and derivatives can depict the memory and inherent process. In 2021, Jin,T. and Xia, H. derived a more fine-grained portrayal of the real economic market based upon the uncertain fractional-order differential equation and put forward European lookback option pricing formulas [15]. Xin-Jiang He and Sha Lin (2021) introduced the stochastic volatility into the finite moment log-stable model, then calculated the analytical formula which is used to price the European options [16].

According to the research results of many international scholars, it is not difficult to find that although there have been many impressive results in the field of option pricing research, few scholars have taken the long memory in the option market and the fractional Brownian motion of the option price's process into account simultaneously. In order to describe both the noise memory and the trend memory effect of stock price, in this paper, we established a fractional order stochastic differential equations which is driven by fractional Brownian motion and apply it to the options pricing. We assume the financial asset price $S$ follows $\mathrm{d}^{\alpha}S = \mu(S,t)\mathrm{d}t^{\alpha} + \sigma(S,t)\mathrm{d}B_H(t)$, then derive the Itô lemma and the partial differential equations under the circumstances of the fractional derivative $\alpha \in (0,1]$ and $(1,2)$ respectively. After that, we derive the European call option pricing formula and apply it to the SSE 50ETF's option pricing.

The main contribution of this article is that we derive a new effective model which can describe both the noise memory and the trend memory effect of financial market. In addition, we apply it into a specific domain - European option pricing. Finally, a new option pricing fomula is obtained. In order to test the effect of the fomula, we conducted empirical analysis and obtained satisfactory results. The main difficulty of this paper is how to combine the fractional stochastic differential equation with the fractional Brownian motion so as to give a more effective European option pricing formula.

The rest of this paper is organized as follows. Section 2 gives some basic concepts and theories on the fractional Brownian motion and fractional order stochastic differential equations. And then establishes the fractional order stochastic differential equation in the financial market. In Section 3, based on the proposed stochastic differential equation with fractional order derivative, we give the corresponding Itô formula under the established model and then derive the fractional European option pricing formula. In Section 4, the traditional B-S model and the fractional stochastic differential equation driven by fractional Brownian motion are used to price the options based on an SSE 50ETF. The results of the mean square error comparison show that the new model is better than the traditional B-S model. The conclusions drawn from this study are presented in Section 5.

## 2. Introduction to fractional Brownian motion and fractional order stochastic differential equations

In this section, we first give some preliminaries about the fractional order integration and derivatives, and then give the relevant properties of fractional Brownian motion. Finally, expand them to the fields of the stochastic differential equations. Thus, based on these previous research results, we can construct the generalized the fractional order stochastic differential equation driven by fractional Brownian motion.

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*2.1. Fractional order integration and derivatives*

There exist various of definitions of fractional derivatives. In this paper, we consider these two definitions, which are Riemann-Liouville integral and Caputo derivative [17].

**Definition 1.** $f(x)$ is a continuous function. Its Riemann-Liouville fractional integral of order $\alpha$ of function $f(x)$ is defined as follows:

$$I^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}f(t)dt \quad \alpha>0, x>0 \tag{1}$$

where $\alpha$ is a fraction and $\Gamma(\alpha)$ is the Gamma function with $\Gamma(\alpha)=\int_{0}^{\infty}x^{\alpha-1}\exp(-x)dx$.

**Definition 2.** The Caputo fractional derivative of order $\alpha$ of function $f(x)$ is defined as:

$$\frac{d^{\alpha}f}{dx^{\alpha}}=D^{\alpha}f(x)=I^{m-\alpha}D^{m}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-t)^{m-\alpha-1}f^{(m)}(t)dt \tag{2}$$

where $\alpha$ is a fraction, $m$ is an integer and $m=[\alpha]$ is the value of $\alpha$ rounded up to the nearest integer, and $f^{(m)}$ is the ordinary derivative of $f$.

And the relationship between fractional difference and finite difference is obtained as follows [14]:

$$\begin{aligned} d^{\alpha}f&=\Gamma(1+\alpha)df \quad 0<\alpha\leq 1\\ d^{\alpha}f&=\Gamma(1+\alpha)[df-f'(x)dx] \quad 1<\alpha<2 \end{aligned} \tag{3}$$

For the purpose of constructing the fractional order stochastic differential equations in this section, now we give some results of the integral with respect to $(dt)^{\alpha}$ in Lemma 1 presented below [18].

**Lemma 1.** Let $f(t)$ denote a continuous function, then its integral with respect to $(dt)^{\alpha}$ is defined by the following equalities:

$$\begin{aligned} \int_{0}^{t}f(\tau)(d\tau)^{\alpha}&=\alpha\int_{0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau \quad 0<\alpha<1\\ \int_{0}^{t}f(\tau)(d\tau)^{\alpha}&=\alpha(\alpha-1)\int_{0}^{t}(t-\tau)^{\alpha-2}F(\tau)d\tau \quad 1<\alpha<2 \end{aligned} \tag{4}$$

where $F(t)=\int_{0}^{t}f(\tau)d\tau$, on making $f(\tau)=1$, we can have the result: $\int_{0}^{t}f(\tau)(d\tau)^{\alpha}=t^{\alpha}$.

*2.2. Fractional Brownian motion*

Now we introduce the definition of fractional Brownian motion [19,20]:

**Definition 3.** Suppose $(\Omega,F,P)$ is a complete probability space, the fractional Brownian motion with the Hurst parameter $H$ in space is a Gaussian process that satisfies:

(i) For every $t>0$, we have $B_H(t)=0$ and $E\big(B_H(t)\big)=0$.

(ii) $B_H(t)$ has homogeneous increments.

(iii) $\mathrm{Cov}\big(B_H(t),B_H(s)\big)=\frac{1}{2}\{|t|^{2H}+|s|^{2H}-|t-s|^{2H}\}, s,t\in R_+$, where Hurst index $H$ is a constant, and $H\in(0,1)$.

And fractional Brownian motion has some properties [19]:

**Theorem 1.** Fractional Brownian motion $\{B_H(t),t\geq 0\}$ has self-similarity. For every $H\in(0,1)$ and $\alpha>0$, $\{B_H(\alpha t),t\in R_+\}$and $\{\alpha^H B_H(t),t\in R_+\}$have the distributions with the same dimension.

**Theorem 2.** Fractional Brownian motion $\{B_H(t),t\geq 0\}$ has $p$-degree variation property. Suppose that for every $(s,t)\in[0,1], p>0$, there existent a constant $C\big(C=E(|G|^p), G\sim N(0,1)\big)$, we have: $E|B_H(t)-B_H(s)|^p\leq C|t-s|^{pH}$.

*2.3. Fractional order stochastic differential equation driven by fractional Brownian motion*

Here, we generalize the classic stochastic differential equation to estabish the fractional order stochastic differential equation driven by fractional Brownian motion based on the results presented before and then apply it to the option pricing in the next section.

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**Definition 4.** Assuming that a financial asset price is $S$, according to the fractional order stochastic differential equation, and considering the fractional Brownian motion, we can get the equation as follows:

$$d^{\alpha}S = \mu(S,t)(dt)^{\alpha} + \sigma(S,t)dB_H(t) \tag{5}$$

where $\mu(S,t)$ is the drift parameter, $\sigma(S,t)$ is the diffusion parameter, $dB_H(t) = \epsilon\sqrt{dt^{2H}}, \epsilon \sim N(0,1)$, and $dt$ and $dB_H(t)$ are uncorrelated.

In this paper, we just consider the situation that $\alpha \in (0,2)$. By using the results of (3), we can rewrite (5) into the following form of $dS$ with respect to $(dt)^{\alpha}$ :

$$\begin{aligned} dS &= \frac{\mu(S,t)}{\Gamma(1+\alpha)}(dt)^{\alpha} + \frac{\sigma(S,t)}{\Gamma(1+\alpha)}dB_H(t), \quad 0 < \alpha \le 1 \\ dS &= \frac{\mu(S,t)}{\Gamma(1+\alpha)}(dt)^{\alpha} + \frac{\sigma(S,t)}{\Gamma(1+\alpha)}dB_H(t) + S'(t)dt, \quad 1 < \alpha < 2 \end{aligned} \tag{6}$$

where $S'(t)$ is the first order derivative of $S$ about time $t$.

## 3. European option pricing

In this section, the corresponding Itô formula and European call option pricing formula are derived based on the fractional order stochastic differential equation driven by fractional Brownian motion.

### 3.1. Itô Lemma

First, in order to study the stochastic processes of the martingale type, let's focus on the derivation of Itồ's rule.

**Definition 5.** A continuous semimartingale $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is an adapted process which has the decomposition,

$$X_t = X_0 + M_t + B_H(t); \quad 0 \le t < \infty, \tag{7}$$

where $M = \{M_t, \mathcal{F}_t; 0 \le t < \infty\}, B = \{B_H(t), \mathcal{F}_t; 0 \le t < \infty\}$.

**Lemma 2.** Let $f: \mathbb{R} \to \mathbb{R}$ be a function of class $C^2$ and let $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous semimartingale with decomposition (7), $\langle M \rangle$ is the quadratic variation process of $M$. Then,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_H(s) + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s \tag{8}$$

and equation (8) can be written in differential notation:

$$df(X_t) = f'(X_t)dM_t + f'(X_t)dB_H(t) + \frac{1}{2}f''(X_t)d\langle M\rangle_t \tag{9}$$

**Proof.** Let us fix $t > 0$ and a partition $\Pi = \{t_0, t_1, \dots, t_m\}$ of $[0,t]$, with $0 = t_0 < t_1 < \cdots < t_m = t$. A Taylor expansion yields

$$\begin{aligned} f(X_t) - f(X_0) \quad &= \sum_{k=1}^{m}\{f(X_{t_k}) - f(X_{t_{k-1}})\} \\ &= \sum_{k=1}^{m} f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) + \frac{1}{2}\sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2 \end{aligned} \tag{10}$$

where $\eta_k(\omega) = X_{t_{k-1}}(\omega) + \theta_k(\omega)\left(X_{t_k}(\omega) - X_{t_{k-1}}(\omega)\right)$ for some appropriate $\theta_k(\omega)$ satisfying $0 \le \theta_k(\omega) \le 1, \omega \in \Omega$. We conclude that

$$f(X_t) - f(X_0) = J_1(\Pi) + J_2(\Pi) + \frac{1}{2}J_3(\Pi) \tag{11}$$

where

$$J_1(\Pi) \triangleq \sum_{k=1}^{m} f'\left(X_{t_{k-1}}\right)\left(B_H(t_k) - B_H(t_{k-1})\right)$$
$$J_2(\Pi) \triangleq \sum_{k=1}^{m} f'\left(X_{t_{k-1}}\right)\left(M_{t_k} - M_{t_{k-1}}\right) \tag{12}$$
$$J_3(\Pi) \triangleq \sum_{k=1}^{m} f''\left(\eta_k\right)\left(X_{t_k} - X_{t_{k-1}}\right)$$

It is easily seen that $J_1\Pi$ converges to the Lebesgue-Stieltjes integral $\int_0^t f'(X_s)dB_H(s)$, as the mesh $\| \Pi \| = \max_{1<k<m}|t_k - t_{k-1}|$ of the partition decreases to zero. On the other hand, the process $Y_s(\omega) \triangleq f'\left(X_s(\omega)\right); 0 \le s \le t, \omega \in \Omega$ is in $\mathcal{L}^*$ (adapted,continuous,and bounded); we intend to approximate it by the simple process

$$Y_s^{\Pi}(\omega) \triangleq f'\left(X_0(\omega)\right)1_0(s) + \sum_{k=1}^{m} f'\left(X_{t_{k-1}}(\omega)\right)1_{(t_{k-1},t_k]}(s) \tag{13}$$

Indeed, we have $EI_t^2(Y^{\Pi} - Y) = E\int_0^t |Y_s^{\Pi} - Y_s|^2\, d\langle M\rangle_s \to 0$ as $\| \Pi \| \to 0$, by the bounded convergence theorem, and so

$$J_2(\Pi) = \int_0^t Y_s^{\Pi}\, dM_s \overset{\|\Pi\|\to 0}{\rightarrow} \int_0^t Y_s\, dM_s \tag{14}$$

in quadratic mean.

$J_3(\Pi)$ can be written as

$$J_3(\Pi) = J_4(\Pi) + J_5(\Pi) + J_6(\Pi) \tag{15}$$

where

$$J_4(\Pi) \triangleq \sum_{k=1}^{m} f''\left(\eta_k\right)\left(B_H(t_k) - B_H(t_{k-1})\right)^2$$
$$J_5(\Pi) \triangleq 2\sum_{k=1}^{m} f''\left(\eta_k\right)\left(B_H(t_k) - B_H(t_{k-1})\right)\left(M_{t_k} - M_{t_{k-1}}\right) \tag{16}$$
$$J_6(\Pi) \triangleq \sum_{k=1}^{m} f''\left(\eta_k\right)\left(M_{t_k} - M_{t_{k-1}}\right)^2$$

Because $B_H$ has total variation bounded by $K$, we have

$$|J_4(\Pi)| + |J_5(\Pi)| \le 2K\|f''\|_\infty \left(\max_{1\le k\le m}|B_H(t_k) - B_H(t_{k-1})| + \max_{1\le k\le m}\left|M_{t_k} - M_{t_{k-1}}\right|\right) \tag{17}$$

and thanks to the continuity of the processes $B_H$ and $M$, this last term converges to zero almost surely as $\| \Pi \| \to 0$. As for $J_6(\Pi)$, we define $J_6^*(\Pi) \triangleq \sum_{k=1}^{m} f''\left(X_{t_{k-1}}\right)\left(M_{t_k} - M_{t_{k-1}}\right)^2$ and observe $|J_6^*(\Pi) - J_6(\Pi)| \le V_t^{(2)}(\Pi) \cdot \max_{1\le k\le m}\left|f''(\eta_k) - f''\left(X_{t_{k-1}}\right)\right|$, where $V_t^{(2)}(\Pi)$ is the quadratic variation of $M$ over the partition $\Pi$. According to the Cauchy-Schwarz inequality, we can get

$$E|J_6^*(\Pi) - J_6(\Pi)| \le \sqrt{48K^4}\sqrt{E\left(\max_{1\le k\le m}\left|f''(\eta_k) - f''\left(X_{t_{k-1}}\right)\right|\right)^2} \tag{18}$$

and this is seen to converge to zero as $\| \Pi \| \to 0$ because of the continuity of the process $X$ and the bounded convergence theorem. Thus, in order to establish the convergence of the quadratic variation term $J_3(\Pi)$ to the integral $\int_0^t f''(X_s)d\langle M\rangle_s$ as $\| \Pi \| \to 0$, it suffices to compare $J_6^*(\Pi)$ to the approximating sum

$$J_7(\Pi) \triangleq \sum_{k=1}^{m} f''\left(X_{t_{k-1}}\right)\left(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}\right) \tag{19}$$

Then we obtain

$$\begin{aligned} & E|J_6^*(\Pi) - J_7(\Pi)|^2 \\ = \quad & E\left|\sum_{k=1}^{m} f''\left(X_{t_{k-1}}\right)\left\{\left(M_{t_k} - M_{t_{k-1}}\right)^2 - \left(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}\right)\right\}\right|^2 \\ = \quad & E\left[\sum_{k=1}^{m}\left[f''\left(X_{t_{k-1}}\right)\right]^2\left\{\left(M_{t_k} - M_{t_{k-1}}\right)^2 - \left(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}\right)\right\}^2\right] \\ \leq \quad & 2\|f''\|_\infty^2 \cdot E\left[\sum_{k=1}^{m}\left(M_{t_k} - M_{t_{k-1}}\right)^4 + \sum_{k=1}^{m}\left(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}\right)^2\right] \\ \leq \quad & 2\|f''\|_\infty^2 \cdot E\left[V_t^{(4)}(\Pi) + \langle M\rangle_t \cdot \max_{1\leq k\leq m}\left(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}\right)\right] \end{aligned} \tag{20}$$

The bounded convergence theorem shows that the last term in the preceding equations goes to zero as $\|\ \Pi\ \| \to 0$. Thus, we conclude that

$$J_3(\Pi) \overset{\|\Pi\|\to 0}{\to} \int_0^t f''(X_s)d\langle M\rangle_s \tag{21}$$

If $\left\{\Pi^{(n)}\right\}_{n=1}^{\infty}$ is a sequence of partitions of $[0,t]$ with $\|\Pi^{(n)}\| \overset{n\to\infty}{\to} 0$, then for some subsequence $\left\{\Pi^{(n_k)}\right\}_{k=1}^{\infty}$ we have

$$\begin{aligned} \lim_{k\to\infty} J_1\left(\Pi^{(n_k)}\right) &= \int_0^t f'(X_s)dB_H(s) \\ \lim_{k\to\infty} J_2\left(\Pi^{(n_k)}\right) &= \int_0^t f'(X_s)dM_s \\ \lim_{k\to\infty} J_3\left(\Pi^{(n_k)}\right) &= \int_0^t f''(X_s)d\langle M\rangle_s \end{aligned} \tag{22}$$

Thus, passing to the limit in equation (11), we see that equation (8) holds for each $0 \leq t < \infty$. Until now, the lemma 2 has been proved.

We have the following, multidimensional version of Itô's rule.

**Lemma 3.** Let $\left\{M_t \triangleq \left(M_t^{(1)}, \dots, M_t^{(d)}\right), \mathcal{F}_t; 0 \leq t < \infty\right\}$ be a vector of local martingales in $\mathcal{M}^{c,loc}$, $\left\{B_t \triangleq \left(B_t^{(1)}, \dots, B_t^{(d)}\right), \mathcal{F}_i; 0 \leq t < \infty\right\}$ a vector of adapted process of bounded variation with $B_H(t) = 0$, and set $X_t = X_0 + M_t + B_H(t); 0 \leq t < \infty$, where $X_0$ is an $\mathcal{F}_0$-measurable random vector in $\mathbb{R}^d$. Let $f(t,x): [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be of class $C^{1,2}$. Then,

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \frac{\partial}{\partial t} f(s,X_s)ds + \sum_{i=1}^{d}\int_0^t \frac{\partial}{\partial x_i} f(s,X_s)dB_H^{(i)}(s) \\ &+ \sum_{i=1}^{d}\int_0^t \frac{\partial}{\partial x_i} f(s,X_s)dM_s^{(i)} \\ &+ \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^t \frac{\partial^2}{\partial x_i\,\partial x_j} f\left(s,X(s)\right)d\left\langle M^{(i)},M^{(j)}\right\rangle_s, \quad 0 \leq t < \infty. \end{aligned} \tag{23}$$

To prove this lemma, we only need to follow the proof of lemma 2 procedure above.

Then, we apply the above lemma to the financial market. Assume that $S$ is the stock price which follows the equation $d^{\alpha}S = \mu S(dt)^{\alpha} + \sigma S dB_H(t)$. We have:

When $0 \leq \alpha < 1$. Let $dX_t = dS, dM_s = \frac{\mu S}{\Gamma(1+\alpha)}(ds)^{\alpha}, dB_H(s) = \frac{\sigma S}{\Gamma(1+\alpha)} dB_H(s), \Pi = \{t_0, t_1, \dots, t_m\}$ a partition of $[0, t]$. Thus, according to lemma 2 we obtain

$$\begin{cases} f(S_t) = f(S_0) + \int_0^t f'(S) \frac{\mu S}{\Gamma(1+\alpha)}(ds)^{\alpha} + \int_0^t f'(S) \frac{\sigma S}{\Gamma(1+\alpha)} dB_H(s) + \frac{1}{2}\int_0^t f''(S) d\langle M \\ \langle M \rangle_s = \lim_{\|\Pi\| \to 0} \sum_{k=1}^{m} \left| M_{s_k} - M_{s_{k-1}} \right|^2 \end{cases}$$

Then, after calculate the equation set above, we can get the fomula in differential form $df = \left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)} \frac{\partial^2 f}{\partial S^2} H t^{2H-1}\right] dt + \frac{\mu S}{\Gamma(1+\alpha)} \frac{\partial f}{\partial S}(dt)^{\alpha} + \frac{\sigma S}{\Gamma(1+\alpha)} \frac{\partial f}{\partial S} dB_H(t)$.

When $1 < \alpha < 2$. Let $dX_t = dS, dM_s^{(1)} = \frac{\mu S}{\Gamma(1+\alpha)}(ds)^{\alpha}, dM_s^{(2)} = S'(s)ds, dB_H(s) = \frac{\sigma S}{\Gamma(1+\alpha)} dB_H(s), \Pi = \{t_0, t_1, \dots, t_m\}$ a partition of $[0, t]$. Thus, according to lemma 3 we obtain

$$\begin{cases} \begin{aligned} f(t, S_t) &= f(0, S_0) + \int_0^t \frac{\partial}{\partial t} f(s, S_s) ds + \int_0^t \frac{\partial}{\partial s} f(s, S_s) \frac{\sigma S}{\Gamma(1+\alpha)} dB_H(s) \\ &\quad + \int_0^t \frac{\partial}{\partial s} f(s, S_s) \frac{\mu S}{\Gamma(1+\alpha)} (ds)^{\alpha} + \int_0^t \frac{\partial}{\partial s} f(s, S_s) s'(s) ds \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial s^2} f\left(s, S_(s)\right)\Big) d\left\langle M^{(1)}, M^{(2)} \right\rangle_s \\ d\left\langle M^{(1)}, M^{(2)} \right\rangle_s &= \lim_{\|\Pi\| \to 0} \sum_{k=1}^{m} \left(M_{s_k}^{(1)} - M_{s_{k-1}}^{(1)}\right)\left(M_{s_k}^{(2)} - M_{s_{k-1}}^{(2)}\right) \end{aligned} \end{cases}$$

Then, after calculate the equation set above, we can get the fomula in differential form $df = \left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)} \frac{\partial^2 f}{\partial S^2} H t^{2H-1} + S'(t) \frac{\partial f}{\partial S}\right] dt + \frac{\mu S}{\Gamma(1+\alpha)} \frac{\partial f}{\partial S}(dt)^{\alpha} + \frac{\sigma S}{\Gamma(1+\alpha)} \frac{\partial f}{\partial S} dB_H(t)$.

Finally, we get the following lemma 4.

**Lemma 4.** Assume that the stock price $S$ follows the equation as below:

$$d^{\alpha}S = \mu S(dt)^{\alpha} + \sigma S dB_H(t) \tag{24}$$

Then, the function $f = f(S_t, t)$ is an Itô stochastic process, and the following expressions hold.

When $\alpha \in (0,1]$.

$$df = \left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)} \frac{\partial^2 f}{\partial S^2} H t^{2H-1}\right] dt + \frac{\mu S}{\Gamma(1+\alpha)} \frac{\partial f}{\partial S}(dt)^{\alpha} + \frac{\sigma S}{\Gamma(1+\alpha)} \frac{\partial f}{\partial S} dB_H(t)$$

When $\alpha \in (1,2)$.

$$\begin{aligned} df &= \left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)} \frac{\partial^2 f}{\partial S^2} H t^{2H-1} + S'(t) \frac{\partial f}{\partial S}\right] dt + \frac{\mu S}{\Gamma(1+\alpha)} \frac{\partial f}{\partial S}(dt)^{\alpha} \\ &\quad + \frac{\sigma S}{\Gamma(1+\alpha)} \frac{\partial f}{\partial S} dB_H(t) \end{aligned} \tag{25}$$

To price a European option, we first introduce Lemma 5, which connects the fractional order stochastic differential equations driven by fractional Brownian motion to the partial differential equations.

**Lemma 5.** $f(S(t), t)$ is the solution of the partial differential equations:

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)} \frac{\partial^2 f}{\partial S^2} H t^{2H-1} - rf = 0 \tag{26}$$

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where $f(S(t),t) = f\big(S(t)\big)$.

**Proof.** First make portfolios $\Pi = \Delta S - f$ and $d\Pi = \Delta dS - df$.

(i) In the case of $0 < \alpha \leq 1$,

$$\begin{aligned} d\Pi &= \Delta dS - df \\ &= \Delta\left[\frac{\mu S}{\Gamma(1+\alpha)}(dt)^\alpha + \frac{\sigma S}{\Gamma(1+\alpha)}dB_H(t)\right] - \left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}\frac{\partial^2 f}{\partial S^2}Ht^{2H-1}\right]dt \\ &\quad - \frac{\mu S}{\Gamma(1+\alpha)}\frac{\partial f}{\partial S}(dt)^\alpha - \frac{\sigma S}{\Gamma(1+\alpha)}\frac{\partial f}{\partial S}dB_H(t) \\ &= -\left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}\frac{\partial^2 f}{\partial S^2}Ht^{2H-1}\right]dt + \frac{\mu S}{\Gamma(1+\alpha)}\left(\Delta - \frac{\partial f}{\partial S}\right)(dt)^\alpha + \frac{\sigma S}{\Gamma(1+\alpha)}\left(\Delta - \frac{\partial f}{\partial S}\right)dB_H(t \end{aligned} \tag{27}$$

When $\Delta = \frac{\partial f}{\partial S}$, we can get the riskless asset portfolio

$$d\Pi = \Delta dS - df = -\left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}\frac{\partial^2 f}{\partial S^2}Ht^{2H-1}\right]dt \tag{28}$$

According to the Bellman Equation, the portfolio $\Pi$ is riskless, we have $d\Pi = r\Pi dt$, where $r$ is the riskless rate. Thus, we get the equation $d\Pi = r\Pi dt = -\left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}\frac{\partial^2 f}{\partial S^2}Ht^{2H-1}\right]dt$. Consequently, we obtain the first partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}\frac{\partial^2 f}{\partial S^2}Ht^{2H-1} - rf = 0 \tag{29}$$

(ii) In the case of $1 < \alpha < 2$,

$$\begin{aligned} d\Pi &= \Delta dS - df \\ &= \Delta\left[\frac{\mu S}{\Gamma(1+\alpha)}(dt)^\alpha + \frac{\sigma S}{\Gamma(1+\alpha)}dB_H(t) + S'(t)dt\right] - \left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}\frac{\partial^2 f}{\partial S^2}Ht^{2H-1} + S'(t)\frac{\partial f}{\partial S}\right]dt \\ &\quad - \frac{\mu S}{\Gamma(1+\alpha)}\frac{\partial f}{\partial S}(dt)^\alpha - \frac{\sigma S}{\Gamma(1+\alpha)}\frac{\partial f}{\partial S}dB_H(t) \\ &= -\left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}\frac{\partial^2 f}{\partial S^2}Ht^{2H-1} + S'(t)\frac{\partial f}{\partial S} - \Delta S'(t)\right]dt + \frac{\mu S}{\Gamma(1+\alpha)}\left(\Delta - \frac{\partial f}{\partial S}\right)(dt)^\alpha \\ &\quad + \frac{\sigma S}{\Gamma(1+\alpha)}\left(\Delta - \frac{\partial f}{\partial S}\right)dB_H(t) \end{aligned} \tag{30}$$

When $\Delta = \frac{\partial f}{\partial S}$, we can also get the riskless asset portfolio

$$d\Pi = \Delta dS - df = -\left[\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}\frac{\partial^2 f}{\partial S^2}Ht^{2H-1}\right]dt \tag{31}$$

And again, $\Pi$ is riskless, similarly, we can get

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}\frac{\partial^2 f}{\partial S^2}Ht^{2H-1} - rf = 0 \tag{32}$$

### *3.2. European call option pricing*

Before we proceed to price the European call option, we make the assumptions as below:

(i) $r$ is the riskless rate and is a constant

(ii) the tax of the stock exchange is free.

(iii) no arbitrage exist in the market.

(iv) the price of the stock obeys

$$\frac{d^\alpha S}{S} = \mu(dt)^\alpha + \sigma dB_H(t). \tag{33}$$

**Theorem 3.** The stock price at the time T follows the formula below.

When $\alpha \in (0,1]$:

$$\begin{aligned} S_T \quad &= \mathrm{Sexp}\left[\frac{\mu}{\Gamma(1+\alpha)}(T^\alpha - t^\alpha) - \frac{\sigma^2}{2\Gamma^2(1+\alpha)}(T^{2H} - t^{2H}) \right. \\ &\left. + \frac{\sigma}{\Gamma(1+\alpha)}\big(B_H(T) - B_H(t)\big)\right] \end{aligned} \tag{34}$$

When $\alpha \in (1,2)$ :

$$\begin{aligned} S_T \quad &= \mathrm{Sexp}\left[\frac{\mu}{\Gamma(1+\alpha)}(T^\alpha - t^\alpha) - \frac{\sigma^2}{2\Gamma^2(1+\alpha)}(T^{2H} - t^{2H}) + \int_t^T \mu\,(S)dS \right. \\ &\left. + \frac{\sigma}{\Gamma(1+\alpha)}\big(B_H(T) - B_H(t)\big)\right] \end{aligned}$$

**Proof.** (i)When $0 < \alpha \leq 1$, according to the formula (25), we can get

$$d(\ln S) = \frac{\mu}{\Gamma(1+\alpha)}(dt)^\alpha - \frac{\sigma^2}{\Gamma^2(1+\alpha)} H t^{2H-1} dt + \frac{\sigma}{\Gamma(1+\alpha)} dB_H(t) \tag{35}$$

Use formula (4) to integrate it, we can get

$$\begin{aligned} &S_T \\ = \mathrm{Sexp}&\left[\frac{\mu}{\Gamma(1+\alpha)}(T^\alpha - t^\alpha) - \frac{\sigma^2}{2\Gamma^2(1+\alpha)}(T^{2H} - t^{2H}) + \frac{\sigma}{\Gamma(1+\alpha)}\big(B_H(T) - B_H(t)\big)\right] \end{aligned} \tag{36}$$

(ii) When $1 < \alpha < 2$, according to the formula (25), we can get

$$d(\ln S) = \frac{\mu}{\Gamma(1+\alpha)}(dt)^\alpha + \left[\frac{S'}{S} - \frac{\sigma^2}{\Gamma^2(1+\alpha)} H t^{2H-1}\right] dt + \frac{\sigma}{\Gamma(1+\alpha)} dB_H(t) \tag{37}$$

Among which $\left(\frac{S'}{S}\right) dt = d(\ln S) = \ln S_{t+1} - \ln S_t = m(t), m(t)$ represents the daily logarithm returns of stock S, and $m(t) = \mu(t)dt$. Thus, $\frac{S'}{S} = \mu(t)$. So the formula (37) can be rewritten as

$$d(\ln S) = \frac{\mu}{\Gamma(1+\alpha)}(dt)^\alpha + \left[\mu(t) - \frac{\sigma^2}{\Gamma^2(1+\alpha)} H t^{2H-1}\right] dt + \frac{\sigma}{\Gamma(1+\alpha)} dB_H(t) \tag{38}$$

Similarly, use formula (4) to integrate it, we can get

$$S_T = \mathrm{Sexp}\left[\frac{\mu}{\Gamma(1+\alpha)}(T^\alpha - t^\alpha) - \frac{\sigma^2}{2\Gamma^2(1+\alpha)}(T^{2H} - t^{2H}) + \int_t^T \mu\,(S)dS + \frac{\sigma}{\Gamma(1+\alpha)}\big(B_H(T) - B_H(t)\big)\right] \tag{39}$$

Now, in the following work, we will derive the option pricing formula based on the risk-neutral assumption, so the expected rate of return $\mu$ is equal to the risk-free rate of interest $r$. By solving equation (26), the pricing formula of European call option is given.

**Theorem 4.** The European call option pricing formula is given as below:

$$\begin{aligned} c \quad &= e^{-r(T-t)} E[\max(S_T - K, 0)] \\ &= S(t)N(d_1) - Ke^{-r(T-t)} N(d_2) \end{aligned} \tag{40}$$

Where

$$d_1 = \frac{\ln\frac{S}{K} + r(T-t) + \frac{\sigma^2(T^{2H} - t^{2H})}{2\Gamma^2(1+\alpha)}}{\frac{\sigma\sqrt{T^{2H} - t^{2H}}}{\Gamma(1+\alpha)}}, d_2 = \frac{\ln\frac{S}{K} + r(T-t) - \frac{\sigma^2(T^{2H} - t^{2H})}{2\Gamma^2(1+\alpha)}}{\frac{\sigma\sqrt{T^{2H} - t^{2H}}}{\Gamma(1+\alpha)}} \tag{41}$$

**Proof.** Let $\xi = \ln S, W = fe^{\beta(t)}, \eta = \xi + \alpha(t), \tau = \rho(t)$. Because $\frac{\partial f}{\partial S} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial S} = \frac{1}{S}\frac{\partial f}{\partial \xi}$, so $S\frac{\partial f}{\partial S} = \frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta} = e^{-\beta t}\frac{\partial W}{\partial \eta}, S^2\frac{\partial^2 f}{\partial S^2} = \frac{\partial^2 f}{\partial \xi^2} - \frac{\partial f}{\partial \xi}$. And as $\frac{\partial^2 f}{\partial \xi^2} = \frac{\partial^2 f}{\partial \eta^2} = e^{-\beta(t)}\frac{\partial^2 W}{\partial \eta^2}$, so $S^2\frac{\partial^2 f}{\partial S^2} = e^{-\beta(t)}\frac{\partial^2 W}{\partial \eta^2} - e^{-\beta(t)}\frac{\partial W}{\partial \eta}$.

Thus, we can rewrite (26) as following:

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$$\frac{H\sigma^2 t^{2H-1}}{\Gamma^2(1+\alpha)}\frac{\partial^2 W}{\partial\eta^2}+\frac{\partial W}{\partial\tau}\rho'(t)+\left[\alpha'(t)-\frac{H\sigma^2 t^{2H-1}}{\Gamma^2(1+\alpha)}+r\right]\frac{\partial W}{\partial\eta}-[\beta'(t)+r]W=0 \tag{42}$$

Let

$$\begin{cases}\alpha'(t)-\dfrac{H\sigma^2 t^{2H-1}}{\Gamma^2(1+\alpha)}+r=0\\ \beta'(t)+r=0\\ \rho'(t)=\dfrac{-2Ht^{2H-1}}{\Gamma^2(1+\alpha)}\end{cases}$$

We can easily get

$$\begin{cases}\alpha(t)=r(T-t)-\dfrac{\sigma^2}{2\Gamma^2(1+\alpha)}(T^{2H}-t^{2H})\\ \beta(t)=r(T-t)\\ \rho(t)=\dfrac{T^{2H}-t^{2H}}{\Gamma^2(1+\alpha)}\end{cases}$$

Thus, we have

$$\begin{cases}\dfrac{\partial W}{\partial\tau}&=\dfrac{1}{2}\sigma^2\dfrac{\partial^2 W}{\partial\eta^2}\\ W|_{\tau=0}&=(e^{\eta}-K)^+\end{cases}$$

By using the Poisson formula

$$W=\frac{1}{\sigma\sqrt{2\pi\rho(t)}}\int_{-\infty}^{+\infty}\left(e^{\xi}-K\right)^+e^{-\frac{(\eta-\xi)^2}{2\sigma^2\rho(t)}}d\xi \tag{43}$$

We have

$$\begin{aligned}f&=We^{-\beta(t)}=e^{-r(T-t)}\frac{\Gamma(1+\alpha)}{\sigma\sqrt{2\pi(T^{2H}-t^{2H})}}\int_{e^{\xi}-K}\left(e^{\xi}-K\right)e^{-\frac{(\eta-\xi)^2\Gamma(1+\alpha)}{2\sigma^2(T^{2H}-t^{2H})}}d\xi\\ &=e^{-r(T-t)}\frac{\Gamma(1+\alpha)}{\sigma\sqrt{2\pi(T^{2H}-t^{2H})}}\int_{\ln K}^{+\infty}\left(e^{\xi}-K\right)e^{-\frac{(\eta-\xi)^2\Gamma(1+\alpha)}{2\sigma^2(T^{2H}-t^{2H})}}d\xi\\ &=e^{-r(T-t)}\frac{\Gamma(1+\alpha)}{\sigma\sqrt{2\pi(T^{2H}-t^{2H})}}\left[\int_{\ln K}^{+\infty}e^{\xi-\frac{(\eta-\xi)^2\Gamma(1+\alpha)}{2\sigma^2(T^{2H}-t^{2H})}}d\xi-\int_{\ln K}^{+\infty}K\,e^{-\frac{(\eta-\xi)^2\Gamma(1+\alpha)}{2\sigma^2(T^{2H}-t^{2H})}}d\xi\right]\end{aligned} \tag{44}$$

Now let

$$\begin{cases}z&=\eta-\xi-r(T-t)+\dfrac{T^{2H}-t^{2H}}{2\Gamma^2(1+\alpha)}\\ d_1&=\dfrac{\ln\dfrac{S}{K}+r(T-t)+\dfrac{\sigma^2(T^{2H}-t^{2H})}{2\Gamma^2(1+\alpha)}}{\dfrac{\sigma\sqrt{T^{2H}-t^{2H}}}{\Gamma(1+\alpha)}}\\ d_2&=\dfrac{\ln\dfrac{S}{K}+r(T-t)-\dfrac{\sigma^2(T^{2H}-t^{2H})}{2\Gamma^2(1+\alpha)}}{\dfrac{\sigma\sqrt{T^{2H}-t^{2H}}}{\Gamma(1+\alpha)}}\end{cases}$$

and

$$f=e^{\xi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1}e^{-\frac{z^2}{2}}dz-Ke^{-r(T-t)}\int_{-\infty}^{d_2}e^{-\frac{z^2}{2}}dz \tag{45}$$

Substitute them into formula (44), we can finally get:

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$$f(S(t),t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{46}$$

Overall, we obtain the European option pricing fomula by the fractional order stochastic differential equation driven by fractional Brownian motion.

## 4. Empirical research based on SSE 50ETF

In this section, we select one of the SSE 50ETF options to test our model, and let it compared with the traditional Black-Scholes model.

### *4.1. Data presentation and preprocessing*

In order to compare the fractional order stochastic differential equation driven by fractional Brownian motion with the traditional B-S model, we chose a SSE 50ETF option listed on August 24, 2023 and expiring on October 25, 2023, with a strike price of 2.65. The main reason of choosing this dataset is that SSE 50ETF is the first exchange option product listed in China, and it is one of the most important options, which has research value and practical significance. The mean square error (MSE) is obtained by comparing the theoretical price and the real market price under two pricing models, which can be used to measure the accuracy of the pricing model.

Firstly, we need to find out the current price of the underlying assets of the SSE 50ETF. The closing prices of SSE 50ETF between August 24,2023 and October 25,2023 are shown in the table 1 below.

**Table 1.** Closing price of the underlying asset of SSE 50ETF (Unit: Yuan)

| Time | $S_t$ | Time | $S_t$ | Time | $S_t$ |
|---|---|---|---|---|---|
| 0 | 2.553 | 13 | 2.609 | 26 | 2.575 |
| 1 | 2.563 | 14 | 2.601 | 27 | 2.559 |
| 2 | 2.596 | 15 | 2.608 | 28 | 2.567 |
| 3 | 2.606 | 16 | 2.596 | 29 | 2.591 |
| 4 | 2.599 | 17 | 2.610 | 30 | 2.566 |
| 5 | 2.590 | 18 | 2.609 | 31 | 2.543 |
| 6 | 2.610 | 19 | 2.597 | 32 | 2.553 |
| 7 | 2.658 | 20 | 2.572 | 33 | 2.543 |
| 8 | 2.641 | 21 | 2.634 | 34 | 2.474 |
| 9 | 2.637 | 22 | 2.609 | 35 | 2.460 |
| 10 | 2.611 | 23 | 2.592 | 36 | 2.441 |
| 11 | 2.596 | 24 | 2.604 | 37 | 2.449 |
| 12 | 2.613 | 25 | 2.583 | 38 | 2.461 |

Where $S_t$ represents the closing price of the underlying asset of SSE 50ETF.

Then, we need to calculate the risk-free interest rate. We obtain one-year Shanghai Interbank Offered Rate on Shibor's official website as the reference rate for calculation, which showed in table 2.

The arithmetic mean is $\bar{r} = (\sum_{t=0}^{38} r_t)/39 = 2.38551\%$. Then convert ordinary compound interest into continuous compound interest, that is, the risk-free continuous compound interest rate is $r = \ln(1+\bar{r}) = 2.3575\%$.

Then, it's time to calculate the SSE 50ETF's volatility. The daily logarithmic return is written as $R_t$, where $R_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$. We supose that the first two days of this period have the same daily logarithmic return. And the sample variance of daily logarithmic return is defined as $\sigma^2 = \frac{1}{39-1}\sum_{t=1}^{38}(R_t - u)^2 = 0.0000794$, where $u = (\sum_{t=1}^{39} R_t)/39 = -0.000840822$. Then we can easily get the sample standard deviation $\sigma = 0.008908726$, which is the daily volatility of the SSE 50ETF. Since there will be 242 trading days in 2023, its annual volatility is $\sigma_y = \sigma \times \sqrt{242} = 0.138587255$.

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**Table 2.** Shanghai Interbank Offered Rate (2023.8.24-2023.10.25)

| Time | $r_t(\%)$ | Time | $r_t(\%)$ | Time | $r_t(\%)$ |
|---|---|---|---|---|---|
| 0 | 2.256 | 13 | 2.331 | 26 | 2.447 |
| 1 | 2.257 | 14 | 2.339 | 27 | 2.452 |
| 2 | 2.262 | 15 | 2.347 | 28 | 2.457 |
| 3 | 2.270 | 16 | 2.361 | 29 | 2.463 |
| 4 | 2.276 | 17 | 2.374 | 30 | 2.466 |
| 5 | 2.280 | 18 | 2.388 | 31 | 2.474 |
| 6 | 2.283 | 19 | 2.402 | 32 | 2.482 |
| 7 | 2.285 | 20 | 2.409 | 33 | 2.490 |
| 8 | 2.287 | 21 | 2.412 | 34 | 2.497 |
| 9 | 2.295 | 22 | 2.416 | 35 | 2.502 |
| 10 | 2.304 | 23 | 2.432 | 36 | 2.504 |
| 11 | 2.309 | 24 | 2.440 | 37 | 2.507 |
| 12 | 2.321 | 25 | 2.446 | 38 | 2.512 |

Besides, we need to calculate Hurst index. By using the R/S analysis approach, the calculation of the Hurst index consists of the following steps. First, divide the time series into several equal-length subintervals, and calculate the cumulative deviation, range, and standard deviation for each subinterval. Then, calculate the rescaled range for each subinterval, which is the ratio of the range to the standard deviation. Next, take the logarithm of each subinterval and perform a linear regression. The slope of the regression line is the Hurst estimate. In this data, the Hurst index $H = 0.597666$, which means that the sequence has a long-dependence memory.

### *4.2. Model results and comparison*

The European call option is priced with B-S model and the fractional order stochastic differential equation driven by fractional Brownian motion respectively. Since the fractional derivative $\alpha$ and Hurst index are not necessarily related, $\alpha = \frac{1}{2}H, \alpha = H, \alpha = 2H$ are chosen here to price options. The model results are shown in table 3.

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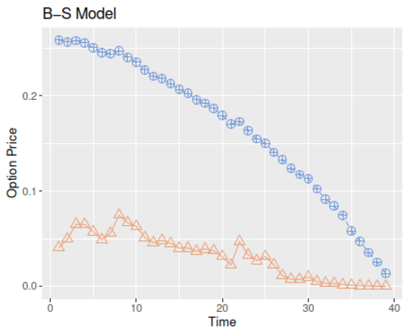
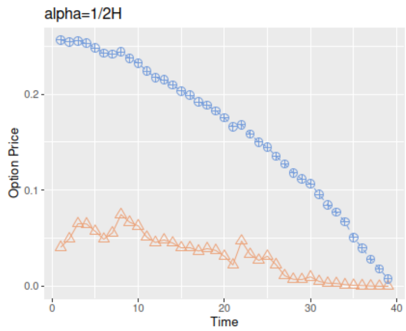
**Table 3.** Model results and comparison

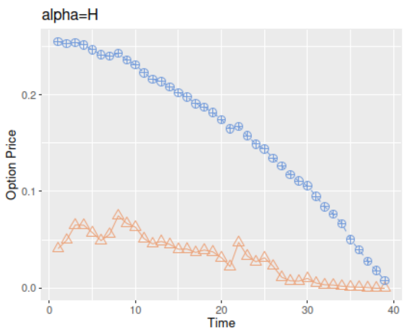
| Time | $C$ | $B-S$ | $\alpha=\frac{1}{2}H$ | $\alpha=H$ | $\alpha=2H$ |
|---|---|---|---|---|---|
| 0 | 0.0407 | 0.2585 | 0.2564 | 0.2553 | 0.3066 |
| 1 | 0.0495 | 0.2564 | 0.2542 | 0.2531 | 0.3048 |
| 2 | 0.0652 | 0.2576 | 0.2553 | 0.2542 | 0.3062 |
| 3 | 0.0650 | 0.2554 | 0.2529 | 0.2518 | 0.3042 |
| 4 | 0.0573 | 0.2505 | 0.2480 | 0.2469 | 0.2996 |
| 5 | 0.0491 | 0.2452 | 0.2427 | 0.2416 | 0.2946 |
| 6 | 0.0556 | 0.2441 | 0.2414 | 0.2403 | 0.2936 |
| 7 | 0.0747 | 0.2470 | 0.2441 | 0.2430 | 0.2964 |
| 8 | 0.0668 | 0.2402 | 0.2373 | 0.2362 | 0.2899 |
| 9 | 0.0630 | 0.2352 | 0.2323 | 0.2312 | 0.2851 |
| 10 | 0.0511 | 0.2270 | 0.2240 | 0.2229 | 0.2771 |
| 11 | 0.0457 | 0.2203 | 0.2172 | 0.2161 | 0.2704 |
| 12 | 0.0483 | 0.2180 | 0.2148 | 0.2137 | 0.2681 |
| 13 | 0.0451 | 0.2126 | 0.2094 | 0.2083 | 0.2627 |
| 14 | 0.0401 | 0.2066 | 0.2032 | 0.2021 | 0.2565 |
| 15 | 0.0403 | 0.2025 | 0.1990 | 0.1979 | 0.2523 |
| 16 | 0.0368 | 0.1957 | 0.1920 | 0.1910 | 0.2452 |
| 17 | 0.0393 | 0.1923 | 0.1885 | 0.1874 | 0.2415 |
| 18 | 0.0374 | 0.1867 | 0.1827 | 0.1817 | 0.2355 |
| 19 | 0.0313 | 0.1794 | 0.1753 | 0.1743 | 0.2279 |
| 20 | 0.0223 | 0.1703 | 0.1661 | 0.1651 | 0.2184 |
| 21 | 0.0471 | 0.1728 | 0.1682 | 0.1672 | 0.2198 |
| 22 | 0.0332 | 0.1633 | 0.1586 | 0.1576 | 0.2098 |
| 23 | 0.0271 | 0.1548 | 0.1500 | 0.1490 | 0.2006 |
| 24 | 0.0313 | 0.1500 | 0.1449 | 0.1439 | 0.1947 |
| 25 | 0.0226 | 0.1406 | 0.1354 | 0.1344 | 0.1844 |
| 26 | 0.0111 | 0.1328 | 0.1273 | 0.1264 | 0.1754 |
| 27 | 0.0074 | 0.1238 | 0.1181 | 0.1172 | 0.1651 |
| 28 | 0.0071 | 0.1176 | 0.1116 | 0.1107 | 0.1573 |
| 29 | 0.0099 | 0.1130 | 0.1067 | 0.1059 | 0.1509 |
| 30 | 0.0051 | 0.1022 | 0.0958 | 0.0950 | 0.1383 |
| 31 | 0.0033 | 0.0915 | 0.0848 | 0.0840 | 0.1255 |
| 32 | 0.0030 | 0.0843 | 0.0773 | 0.0766 | 0.1158 |
| 33 | 0.0018 | 0.0744 | 0.0672 | 0.0666 | 0.1032 |
| 34 | 0.0011 | 0.0580 | 0.0508 | 0.0502 | 0.0838 |
| 35 | 0.0008 | 0.0473 | 0.0399 | 0.0394 | 0.0694 |
| 36 | 0.0004 | 0.0356 | 0.0282 | 0.0278 | 0.0535 |
| 37 | 0.0001 | 0.0254 | 0.0183 | 0.0180 | 0.0383 |
| 38 | 0.0001 | 0.0137 | 0.0078 | 0.0076 | 0.0205 |



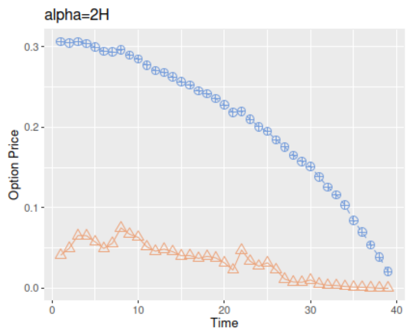

**(a)** Option pricing based on B-S Model




**(b)** Option pricing based on fractional order stochastic differential equations driven by fractional Brownian motion($\alpha = \frac{1}{2}H$)




**(c)** Option pricing based on fractional order stochastic differential equations driven by fractional Brownian motion($\alpha = H$)




**(d)** Option pricing based on fractional order stochastic differential equations driven by fractional Brownian motion($\alpha = 2H$)

**Figure 1.** Visualization results of different models

According to the figure 1, we can easily find that:

(i) Regardless of the traditional pricing model or the newly established model, there are some differences between the empirical results and the real value.

(ii) When $\alpha = 2H$, the order of magnitude of the error is significantly higher than in the other cases.

(iii) It's really hard to judge which model has the best effect precisely.

In order to compare the accuracy of the models more precisely, we use MSE to evaluate it.

$$MSE = \frac{\sum_{i=1}^{n}\left(\hat{C}_i - C_i\right)^2}{n} \tag{55}$$

where $C_i$ is the real option price in day $i$ while $\hat{C}_i$ is the estimated option price in day $i$.

Their MSEs are calculated respectively, and the results are shown in the table 4 below.

**Table 4.** Model results and comparison

| Model | MSE | Model | MSE |
|---|---|---|---|
| B-S | 0.02092517 | $\alpha = H$ | 0.01963943 |
| $\alpha = \frac{1}{2}H$ | 0.01990621 | $\alpha = 2H$ | 0.03525325 |

According to the results, we can draw a conclusion that the pricing effect of European option based on fractional stochastic differential equation driven by fractional Brownian motion is better than that of traditional B-S pricing model. In this data with long-dependence memory, the new model has the best simulation effect when $\alpha = H$, while when $\alpha = 2H$, the effect is relatively poor compared with the traditional B-S model. Thus, we can guess the new model has the best effect when $\alpha = H$. In general, European option pricing based on fractional stochastic differential equations driven by fractional Brownian motion is feasible and effective.

## 5. Conclusions and future research

Because the fractional order ordinary differential equations can capture the memory effect in financial system, and fractional Brownian motion can better describe stock prices, we established the fractional order stochastic differential equation by adding the stochastic process into the fractional ordinary differential equation. Based on this equation, we apply the fractional order stochastic differential equation driven by fractional Brownian motion to the financial market. We constructed the stock price $\mathrm{d}^{\alpha}S = \mu(S,t)\mathrm{d}t^{\alpha} + \sigma(S,t)\mathrm{d}B_H(t)$, and derive the stock price process in the case of $0 < \alpha \leq 1$ and $1 < \alpha < 2$, respectively, and the European call option pricing formula under the fractional order stochastic differential equation driven by fractional Brownian motion. After empirical research, it is obvious that the new model has better results, which proves that the given European option pricing by fractional stochastic differential equation driven by fractional Brownian motion is feasible and effective.

It would be an interesting work if we improve our model by connecting fractional order stochastic differential equation with the mixed fractional jump diffusion environment, which can price the barrier options, helping investors grasp the risks related to barrier options more intuitively and effectively.

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